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PREFACE

In this book the authors for the first time introduce a new type of topological spaces called the set ideal topological spaces using rings or semigroups, or used in the mutually exclusive sense. This type of topological spaces use the class of set ideals of a ring (semigroups). The rings or semigroups can be finite or infinite order.

By this method we get complex modulo finite integer set ideal topological spaces using finite complex modulo integer rings or finite complex modulo integer semigroups. Also authors construct neutrosophic set ideal toplogical spaces of both finite and infinite order as well as complex neutrosophic set ideal topological spaces.

Several interesting properties about them are defined, developed and discussed in this book.

The authors leave it as an open conjecture whether the number of finite topological spaces built using finite sets is increased by building these classes of set ideal topological spaces using finite rings or finite semigroups.

It is to be noted for a given finite semigroup or a finite ring we can have several number of set ideal topological spaces using different subsemigroups or subrings.

The finite set ideal topological spaces using a semigroup or a finite ring can have a lattice associated with it. At times these lattices are Boolean algebras and in some cases they are lattices which are not Boolean algebras.

Minimal set ideal topological spaces, maximal set ideal topological spaces, prime set ideal topological spaces and S-set ideal topological spaces are defined and studied.

Each chapter is followed by a series of problems some of which are difficult and others are routine exercises.

This book is organized into four chapters. First chapter is introductory in nature. Set ideals in rings and semigroups are developed in chapter two. Chapter three introduces the notion of set ideal topological spaces. The final section gives some more classes of set ideal topological spaces and they can be applied in all places where topological spaces are applied under constraints and appropriate modifications.

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W.B.VASANTHA KANDASAMY
FLORENTIN SMARANDACHE

Chapter One

INTRODUCTION

In this chapter we just mention the concepts used in this book by giving only the references where they are available so that the reader who is not aware / familier with it can refer them.

Throughout this book Z is the set of integers. Z_n the modulo integers modulo n; $1 < n < \infty$; Q the field of rationals, R the reals, C the complex field. $C(Z_n) = \{a + bi_F \mid a, b \in Z_n, i_F^2 = n-1\}$, the ring of finite complex modulo integers, $\langle R \cup I\rangle$; the real neutrosophic numbers, $\langle Q \cup I\rangle$, the rational neutrosophic numbers, $\langle Z \cup I\rangle$, the integers neutrosophic numbers, $\langle Z_n \cup I\rangle = \{a + bI \mid a, b \in Z_n, I^2 = I\}$ the modulo integer neutrosophic numbers, $\langle C \cup I\rangle = \{a + bI \mid$ a and b are complex numbers of the form x + iy, t + is, x, y, t, s reals and $i^2 = -1\}$ the neutrosophic complex numbers and $\langle C(Z_n) \cup I\rangle = \{a + bi_F + cI + di_FI \mid a, b, c, d \in Z_n, I^2 = I, i_F^2 = n-1, (i_FI)^2 = (n-1)I\}$ the neutrosophic finite complex modulo integers.

We denote by S a semigroup only under multiplication. S(n) denotes the symmetric semigroup of mappings of a set

(1, 2, …, n) to itself. Let R be any ring finite or infinite, commutative or non commutative. I an ideal of R (I is a subring of the ring R and for all x ∈ I and a ∈ R xa and ax ∈ I) [7]. Let S be a semigroup. A ⊆ S, a proper subset of S is an ideal if

(i) A is a subsemigroup
(ii) for all a ∈ A and s ∈ S as and sa ∈ A.

We also define Smarandache semigroup (S-semigroup), Smarandache subsemigroup(S-subsemigroup) and Smarandache ideal (S-ideal) [6-7].On similar lines Smarandache structures are defined for rings.

We use the notion of lattices and Boolean algebras [2]. Further the concept of topological space is used [1, 5]. We use the notion of dual numbers, special dual like numbers and special quasi dual numbers both of finite and infinite order rings [9-11].

For finite complex modulo numbers and finite neutrosophic complex modulo integers refer [8, 13].

Chapter Two

SET IDEALS IN RINGS

In this chapter for the first time we introduce the notion of set ideals in rings, a new substructure in rings. Throughout this chapter R, denotes a commutative ring or a non commutative ring. We define set ideals of a ring and compare them with ideals and illustrate it by examples.

Interrelations between ideals and set ideals are brought out. These notions would be helpful for we see union of subrings are not subrings in general. Using these concepts we can give some algebraic structure to them; like lattice of set ideals and set ideal topological space of a ring over a subring.

Here we proceed onto introduce the notion of ideals in rings and illustrate them by examples.

DEFINITION 2.1: *Let R be a ring. P a proper subset of R. S a proper subring of R ($S \neq R$). P is called a set left ideal of R relative to the subring S of R if for all $s \in S$ and $p \in P$, $sp \in P$.*

One can similary define a set right ideal of a ring R relative to the subring S of R.

A set ideal is thus simultaneously a left and a right set ideal of R relative to the subring S of R.

We shall illustrate this by some simple examples.

Example 2.1: Let $R = Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the ring of integers modulo 10. Take $P = \{0, 5, 6\} \subseteq R$, P is a set ideal of R relative to the subring $S = \{0, 5\}$.

Clearly P is not a set ideal of R relative to the subring $S_1 = \{0, 2, 4, 6, 8\}$.

Example 2.2: Consider the ring of integers modulo 12, $Z_{12} = \{0, 1, 2, 3, \ldots, 11\}$. Take $S = \{0, 6\}$ a subring of Z_{12}. $P = \{0, 2, 4, 8, 10\} \subseteq Z_{12}$. P is a set ideal of Z_{12} relative the subring $S = \{0, 6\}$. Clearly P is not a set ideal of Z_{12} relative to the subring $S_1 = \{0, 3, 6, 9\}$.

P is a set ideal of Z_{12} relative to the subrings $S_3 = \{0, 4, 8\}$ and $S_2 = \{0, 2, 4, 6, 8, 10\}$.

Now we make the following proposition.

Proposition 2.1: *Let R be a ring. If $P \subseteq R$ is a set ideal over a subring S of R then P in general need not be a set ideal relative to every other subring of R.*

***Proof*:** We can prove this only by counter example. Example 2.1 proves the assertion. Hence the result.

THEOREM 2.1: *Let R be any ring, {0} is the set ideal of R relative to every subring S of R.*

***Proof*:** Clear from the definition.

THEOREM 2.2: *Let R be a ring with unit. The set P = {1} is never a set ideal of R relative to any subring S of R.*

***Proof*:** Clearly by the very definition for if S is any subring of R. P = {1} is a set ideal of R. S.P = P.S $\neq$ {1} but is only S i.e., SP = PS = S.

Hence the claim.

THEOREM 2.3: *Every ideal of a ring R is a set ideal of R for every subring S of R.*

Proof: Let I be any ideal of R. Clearly for every subring S of R we see I is a set ideal of R for every subring S or R we have SI $\subset$ I and IS $\subset$ I.

THEOREM 2.4: *In the ring of integers Z there exists no finite set {0} $\neq$ P $\subseteq$ Z which is a set ideal of R for any subring S of Z.*

***Proof*:** Clearly every subring of Z is of infinite cardinality. So if P is any finite set SP $\subseteq$ P is an impossibility as Z is an integral domain with no zero divisors. Since SP $\neq$ P for any subring S of Z, we see no finite subset of Z is a set ideal of Z relative to any subring of Z as every subring of Z is of infinite order.

Hence the claim.

We see as in case of ideals we cannot define the notion of principal set ideals of R relative to any subring S of R. Infact {0} is the only set principal ideal of R.

Now we proceed onto define the notion of set prime ideal of R.

DEFINITION 2.2: *Let R be any ring. Suppose P $\subset$ R is a set ideal of R relative to the subring S of R and if x = p.q $\in$ P then p or (and) q is in P. We call only such set ideals to be prime set ideals.*

We illustrate this situation by some examples.

Example 2.3: Let $Z_{10} = \{0, 1, 2, \ldots, 9\}$ be a ring of integers modulo 10. Let $P = \{0, 5, 2, 6\} \subseteq Z_{10}$. Clearly P is a set prime ideal of R relative to the subring $S = \{0, 5\}$ of R. The elements of P need not be in the subring S. Now consider the set $P_1 = \{0, 5, 6\} \subseteq Z_{10}$, $6 = 2.3 \in P_1$ but 3 and $2 \notin P_1$ so P_1 is not a set prime ideal of R relative to the subring $S = \{0, 5\} \subseteq Z_{10}$.

THEOREM 2.5: *Let R be a ring. S a subring of R. If $P \subseteq R$ is a set ideal of R relative to the subring S of R, then $0 \in P$.*

Proof: Follows from the simple fact $0 \in S$ (as S is a subring of R) so $0.p = 0 \in P$.

Hence the claim.

Now we proceed onto define the notion of set maximal ideal of a ring R relative to a subring S of R.

DEFINITION 2.3: *Let R be a ring, S a subring of R. P a proper subset not an ideal or subring of R. We say P is a set maximal ideal of R relative to the subring S and if P_1 is another proper subset not a subring or ideal of R such that $P \subset P_1 \subset R$ and P_1 is also set ideal of R relative to the same subring S of R then either $P = P_1$ or $P_1 = R$.*

We illustrate this by some examples.

Example 2.4: Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the ring of integers modulo 6. Let $P = \{4, 3, 5, 0, 2\} \subseteq Z_6$; P is a set maximal ideal of Z_6 relative to the subring $S = \{0, 3\}$. Infact $P \underset{\neq}{\subset} P_1 \subseteq R$ is impossible as $P_1 = R$ is the only possibility.

Infact $P = \{0, 2, 3, 4, 5\} \subseteq Z_6$ is also a set maximal ideal of Z_6 relative to the subring $S_1 = \{0, 2, 4\}$.

In this ring if we take $P_3 = \{0, 5, 3\} \subseteq Z_6$, P_3 is a set ideal of Z_6 relative to the subring $S = \{0, 3\}$, clearly P_3 is not a set maximal ideal of Z_6 relative to the subring S.

From the above observation it is clear that in general every set ideal of a ring R relative to a subring S need not be a set maximal ideal of R relative to a subring S of R.

Now we proceed onto define the notion of set minimal ideals of a ring R relative to the subring S of R.

DEFINITION 2.4: *Let R be a ring. P a proper subset of R. P is said to be a set minimal ideal of R relative to a subring S of R if* $\{0\} \neq P_1 \subseteq P \subset R$ *where P is a set ideal of R relative to the same subring S of R then either* $P_1 = \{0\}$ *or* $P_1 = P$.

Now we illustrate this situation by some examples.

Example 2.5: Let $Z_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be the ring of integers modulos 8. Let $P = \{0, 2\} \subseteq Z_8$. P is a set minimal ideal of the ring Z_8 relative to the subring $S = \{0, 4\} \subseteq Z_8$.

Clearly P is set minimal ideal for if $P_1 \neq \{0\}$ then $P_1 = \{2\}$ but $P_1 = \{2\}$ is not a set minimal ideal of Z_8 relative to the subring $S = \{0, 4\}$ as $0 \notin P_1$ but $4.2 = 0 \pmod 8$.

Hence the claim.

We show by an example in general all set ideals need not be set minimal ideals of R.

Example 2.6: Let $R = Z_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be the ring of integers modulo 12. Take $P = \{0, 2, 8, 10\} \subseteq Z_{12}$. P is a set ideal of R relative to the subring, $S = \{0, 6\} \subseteq Z_{12}$. Clearly P is not a set minimal ideal of R relative to the subring $\{0, 6\} = S$. For take $P_1 = \{0, 2\} \subseteq P$. P_1 is a set ideal of R relative to the subring $S = \{0, 6\}$. Take $P_2 = \{0, 8\} \subseteq P$. P_2 is also a set ideal of R relative to the subring $S = \{0, 6\}$.

If $P_3 = \{0, 10\} \subset P$, P_3 is also a pseudo set ideal of R relative to the same subring $S = \{0, 6\}$.

From this it is evident that all set ideals are not in general minimal.

Now we formulate an interesting observation.

THEOREM 2.6: *Let R be a ring with unit. A set P of R containing the unit is a set ideal of R relative to the subring S of R if P contains S.*

Proof: Let $P \subset R$ such that $1 \in P$ be a set ideal of R relative to the subring S of R. Since $1 \in P$ clearly $S.1 \subseteq P$ as $s.1 \in P$ for every $s \in S$. Hence the claim.

Now a natural question would be; if P is a set ideal of a ring R relative to a subring S of R and suppose $S \subset P$ will $1 \in P$. The answer is no. This is proved by an example.

Example 2.7: Let $Z_{15} = \{0, 1, 2, \ldots, 14\}$ be the ring of integers modulo 15. Let $P = \{0, 2, 5, 10, 3, 6\} \subseteq Z_{15}$ be a set ideal of the ring Z_{15} relative to the subring $S = \{0, 5, 10\}$, $S \subseteq P$, clearly $1 \notin P$. Hence the claim. We see $P \cup \{1\} = P_1$ is also a set ideal of Z_{15} over S.

Note: It is important and interesting to note that unlike an ideal the set ideal can contain one. This is evident from the example 2.7. Still it is interesting to see that a field of characteristic zero can have set ideals. This is impossible in case of usual ideals.

This is explained by the following example.

Example 2.8: Let Q be field of rationals. Clearly $S = 2Z$ is a subring of Q.

Now take $P = \{0, \pm 2n, \pm 3n, \pm 5n, \pm 7n, \pm 9n, \pm 11n\} \subseteq Q$. P is a set ideal of Q relative to the subring $S = 2Z$.

Thus a field of characteristic zero can have set ideals relative to a subring.

THEOREM 2.7: *A prime field of characteristic p, Z_p; p a prime cannot have non trivial set ideals.*

***Proof*:** Z_p be the prime field of characteristic p, p a prime. Clearly P has no proper subrings other than {0}. So even if P is any set P cannot become a set ideal relative to a subring S of R.

We for the first time using the set ideals P of a ring R relative to the subring S of R define the notion of set quotient ideal related to P.

DEFINITION 2.5: *Let R be a ring, P a set ideal of R relative to the subring S of R. i.e., P is only a subset and not a subring of R. Then R/P is the set quotient ideal if and only if R / P is a set ideal of R relative to the same subring S of R.*

We illustrate this by some examples.

Example 2.9: Let Z_{12} = {0, 1, 2, …, 11} be the ring of integers modulo 12. S = {0, 6} be a subring of Z_{12}.

Let P = {0, 2, 5, 8, 10} $\subseteq Z_{12}$ be a proper subset of Z_{12}. Clearly P is not a subring of Z_{12} only a subset of Z_{12}. P is clearly a set ideal of Z_{12} relative to the subring S = {0, 6} of Z_{12}.

$$\frac{Z_{12}}{P} = \{P,\ \overline{1} + P,\ \overline{3} + P,\ \overline{4} + P,\ \overline{6} + P,\ \overline{7} + P,\ \overline{9} + P,\ \overline{11} + P\}.$$

It is easily verified $\frac{Z_{12}}{P}$ is a set quotient ideal relative to the subring S = {0, 6}. The following observations are both interesting and important.

1. $\frac{Z_{12}}{P}$ is not a subset of Z_{12}.
2. For any x ∈ $\frac{Z_{12}}{P}$ we see sx ∈ $\frac{Z_{12}}{P}$ for every s ∈ S, S = {0, 6} is a subring of Z_{12}.

Further in case of ring R happens to be a finite ring we see o(P) + o(R/P) – 1 = number of elements in R by o(P), we mean only the number of elements in P.

We give yet another example of set quotient ideals of a ring R.

Example 2.10: Let Z_{20} = {0, 1, 2, …, 19} be the ring of integers modulo 20. Let S = {0, 5, 10, 15} be a subring of Z_{20}. Take P = {0, 4, 8, 10, 6, 12} $\subseteq Z_{20}$ to be a set ideal of Z_{20} relative to the subring S = {0, 5, 10, 15}.

Consider $\frac{Z_{20}}{P} = \{P,\ \overline{1}+P,\ \overline{2}+P,\ \overline{3}+P,\ \overline{5}+P,\ \overline{7}+P,$

$\overline{9}+P,\ \overline{11}+P,\ \overline{13}+P,\ \overline{14}+P,\ \overline{15}+P,\ \overline{16}+P,\ \overline{17}+P,\ \overline{18}+$

$P,\ \overline{19}+P\}$.

$|P| = 6 \quad o\left(\frac{Z_{20}}{P}\right) = 15.$

$o(Z_{20}) = 20 = 6 + 15 - 1.$

Now we are interested to study the following two properties:

1. Will every ideal I of R always contain a set ideal of R relative to some subring S of R?
2. Will every set ideal of R relative to a subring S of R contain any proper subring of R?

The answer to the first question is no.

For we can give many examples of ideals which do not contain a set ideal relative to some ring S of R. It is sufficient if we illustrate this by an example.

Example 2.11: Let Z_{18} = {0, 1, 2, …, 17} be the ring of integers modulo 18.

Take I = {0, 9} $\subseteq Z_{18}$. I is an ideal of Z_{18}. But I does not contain any proper subset which can be a set ideal of Z_{18} relative to any subring S of Z_{18}.

Take P = {0, 6} $\subseteq Z_{18}$. P is a set ideal of Z_{18} relative to the subring S = {0, 9}. Infact J = {0, 6, 12} is an ideal of Z_{18} which contains a proper set ideal P = {0, 6} $\subseteq$ J in Z_{18} relative to the subring S = {0, 9}.

We call those ideals I of a ring R which do not contain any set ideals P in it as simple set ideals of R.

We will answer the second question. Every set ideal of a ring R relative to a subring S of R need not contain any proper subring S of R.

We illustrate this by the following example.

Example 2.12: Let

Z_{18} = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, …, 16, 17} be the ring of modulo integers 18. Take P = {0, 2, 6, 10, 16} $\subseteq Z_{18}$ to be a set ideal of Z_{18} relative to the subring S = {0, 9} $\subseteq Z_{18}$. It is easily verified P does not contain any proper subring of Z_{18} other than the trivial {0} subring.

We prove the following theorem.

THEOREM 2.8: *If R is a finite ring with 1. P $\subset$ R, a set ideal of R relative to the subring S of R and S $\not\subseteq$ P and S $\cap$ P = {0}; then if R/P is a set quotient ideal then R/P contains 1 + P as its unit; and R/P contains a subring $\overline{S}$ such that $\overline{S} \cong S$.*

Proof: Given R is a finite ring with unity 1. Let P $\subseteq$ R be a proper subset of R which is a set ideal of R relative to the subring S of R. S $\neq$ R. Now the set R/P = {P, 1 + P, …} since 1 $\notin$ P for if 1 $\in$ P then S $\subseteq$ P, a contradiction to our assumption that is S $\not\subseteq$ P and S $\cap$ P = {0}. Now R / P is a set and R/P is

given to be a set quotient ideal relative to the subring S of R. Since $1 + P \in R/P$.

$S + P \subseteq R/P$ as every $s + P \in R/P$ for every $s \in S$ as $P \cap S = \{0\}$.

Thus if $S + P = \overline{S}$, then $\overline{S}$ is a ring isomorphic to the subring S of R. Hence the claim.

Now we define a new notion called Smarandache set ideal of a ring as follows:

DEFINITION 2.6: *Let R be any ring. S a subring of R. Suppose P is a set ideal of R relative to the subring S of R and $S \underset{\neq}{\subset} P$ then we call P to be a Smarandache set ideal of the ring R relative to the subring S of R. In short we call P as S-set ideal of R.*

We will first illustrate this situation by the following examples.

Example 2.13: Let $R = Z_{30} = \{0, 1, 2, \ldots, 29\}$ be the ring of integers modulo 30. Take $S = \{0, 10, 20\} \subseteq Z_{30}$ to be a subring of Z_{30}. Let $P = \{0, 3, 6, 10, 20, 9, 12, 15, 18, 21\} \subseteq Z_{30}$ be a set ideal of the ring R relative to the subring S of $R = Z_{30}$. Clearly $S \subseteq P$ so P is a Smarandache set ideal of $R = Z_{30}$ relative to the subring $S = \{0, 10, 20\} \subseteq R$.

Example 2.14: Let $R = Z_{42} = \{0, 1, 2, \ldots, 11\}$ be the ring integers modulo 12. Let $S = \{0, 6\}$ is subring of R. Take $P = \{0, 6, 3, 2, 4, 8\} \subseteq Z_{12}$. P is a set ideal of R; infact P is a S-set ideal of R.

A natural question would be do we have set ideals of R which are not S-set ideals of R.

THEOREM 2.9: *Let R be a ring. Every Smarandache set ideal of a ring R is a set ideal P of R but a set ideal P of a ring R need not in general be a S-set ideal of R.*

***Proof*:** Suppose P is given to be a Smarandache set ideal of the ring R relative to the subring S of R; then clearly P is a set ideal of R by the very definition of the Smarandache set ideal of the ring R.

Conversely if P is a set ideal of R relative to the subring S of R then P in general need not be a S set ideal of R.

This can be proved only by giving a counter example.

Let Z_{12} = R = {0, 1, 2, …, 11} be the ring of integers modulo 12. Let S = {0, 6} $\subseteq$ Z_{12} be the subring of R. Take P = {0, 2, 4, 8, 9, 10} $\subseteq$ Z_{12}; P is easily verified to be a set-ideal of R relative to the subring S of R. But P is not a Smarandache set ideal of R as S $\not\subseteq$ P. Hence the claim.

Now it may so happen that we have P to be a set ideal of R relative to a subring S of R. But P does not contain S but P contains some other subring S_1 of R. In such cases we make the following definition.

DEFINITION 2.7: *Let R be a ring. Let P be a set ideal of the ring R relative to the subring S of R. Suppose P contains a subring S_1 of R, $S_1 \neq R$ then we call R to be a Smarandache quasi set ideal of R relative to the subring S of R.*

We illustrate this by the following example.

Example 2.15: Let Z_{12} = {0, 1, 2, …, 11} be the ring of integers modulo 12. S = {0, 3, 6, 9} be a subring of Z_{12}. Take P = {0, 4, 8, 10, 6} $\subseteq$ Z_{12}. P is easily verified to be a Smarandache quasi set ideal of Z_{12} relative to the subring S = {0, 3, 6, 9} and S_1 = {0, 4, 8} $\subseteq$ P is a subring of Z_{12}.

Example 2.16: Let Z_{20} = {0, 1, 2, …, 19} be the ring of integers modulo 20. Take S = {0, 5, 10, 15} a subring of Z_{20}.

Take P = {0, 2, 4, 8, 10, 12, 16, 18} $\subseteq Z_{20}$. P is easily seen to be a S-quasi set ideal of Z_{20} as P contains the subring S_1 = {0, 4, 8, 12, 16}; P $\subseteq Z_{20}$.

We now show that every S-quasi set ideal of a ring need not be a Smarandache set ideal and a S-set ideal is not always a S-quasi set ideal. This is only proved using examples in the following.

THEOREM 2.10: *A Smarandache set ideal in general is not a S-quasi set ideal and a S-quasi set ideal in general is not a S-set ideal.*

***Proof*:** We prove this theorem only by giving counter examples.

Take R = Z_{20} = {0, 1, 2, …, 19} to be the ring of integers modulo 20. Let S = {0, 10} be a subring of Z_{20}.

Let P = {0, 4, 8, 12, 16, 6, 18} $\subseteq Z_{20}$. P is not a Smarandache set ideal of R = Z_{20} as S $\not\subseteq$ P. But P is a S-quasi set ideal of R = Z_{20} as P contains the subring S_1 = {0, 4, 8, 12, 16} $\subseteq$ P $\subseteq Z_{20}$.

Hence the claim.

Now take S = Z_{18} = {0, 1, 2, …, 17}; the ring of integers modulo 18. Take S = {0, 6, 12} $\subseteq Z_{18}$ to be a subring.

Let P = {0, 6, 12, 10, 5} $\subseteq Z_{18}$. P is a S-set ideal of R but is not a S-quasi set ideal of R = Z_{20}.

Hence the claim.

Now take Z_{30} = {0, 1, 2, …., 29} the ring of integers modulo 30. Let P = {0, 6, 12, 18, 24, 10, 20, 3, 5} $\subseteq Z_{30}$. Let S = {0, 10, 20} be a subring of R. P is both a S-quasi set ideal of R = Z_{30} relative to S as well as Smarandache set ideal of Z_{30} relative to S as P contains S = {0, 10, 20} and also P contains a subring S_1 = {0, 6, 12, 18, 24}.

Thus we now see that we can take P to be just the set theoretic union of two subrings of a ring R and P may be both S-quasi set ideal as well as S-set ideal of R.

This is one of the advantages of the new definitions for we know union of subrings in general is not a subring, yet we can get a nice structure for it.

We in view of this define a new notion.

DEFINITION 2.8: *Let R be any ring. S and S_1 be two subrings of R. $S \neq S_1$, $S \not\subseteq S_1$ and $S_1 \not\subseteq S$. If P is a subset of R such that P contains both S_1 and S and P is a set ideal of R relative to both S_1 and S then we call P to be a Smarandache strongly quasi set ideal of R.*

We now illustrate this situation by some examples.

Example 2.17: Let $Z_{30} = \{0, 1, 2, \ldots , 29\}$ be the ring of integers modulo 30. Take $S_1 = \{0, 6, 12, 18, 24\}$ and $S = \{0, 10, 20\}$ to be two subring of Z_{30}.

Take $P = \{0, 10, 20, 6, 12, 18, 24, 15\} \subseteq Z_{30}$. P is a set ideal relative to the subring $S = \{0, 20, 10\}$ as well as P is a S set ideal relative to the subring $S_1 = \{0, 6, 12, 18, 24\}$. Thus P is a S-strongly quasi set ideal of R.

We give yet another example.

Example 2.18: Let $Z_{60} = \{0, 1, 2, \ldots, 59\}$ be the ring of integers modulo 60. Take $S_1 = \{0, 10, 20, 30, 40, 50\} \subseteq Z_{60}$ to be a subring of Z_{60}.

Let $S_2 = \{0, 15, 30, 45\} \subseteq Z_{60}$ be another subring of Z_{60}.

Suppose $P = \{0, 10, 20, 30, 40, 50, 15, 30, 45, 8, 4\} \subseteq Z_{60}$. P is a S-strongly quasi set ideal of Z_{60}.

We have the following interesting theorem.

THEOREM 2.11: *Let R be a ring. Every S-strong quasi set ideal P of R is a S-quasi set ideal of R and S-set ideal of R relative to the same subring S of R. But a S-set ideal P of R in general is not a S-strongly quasi set ideal of R. Also a S-quasi set ideal P of R in general is not a S-strongly quasi set ideal of R.*

Proof: We prove these assertions only by examples. Take $R = Z_{60} = \{0, 1, 2, \ldots, 59\}$ the ring of integers modulo 60.

Take $P = \{0, 30, 10, 8\} \subseteq Z_{60}$. P is a S-set ideal of Z_{60} relative to the subring $S = \{0, 30\}$.

For $S = \{0, 30\} \subseteq P = \{0, 30, 10, 8\}$. Clearly P is not a S-strongly quasi set ideal of Z_{60} as P does not even contain any other subring of R other than $S = \{0, 30\}$.

Consider $Z_{30} = \{0, 1, 2, \ldots, 29\}$ be the ring of integers modulo 30. Take $P = \{0, 10, 20, 6, 4, 8\} \subseteq Z_{30}$. P is clearly a S-quasi set ideal of Z_{30} relative to the subring $S = \{0, 15\}$. P contains the subring $S_1 = \{0, 10, 20\} \subseteq Z_{30}$. Thus P is not a S-strongly quasi set ideal of Z_{30} as $S = \{0, 15\} \not\subseteq P$.

Hence the claim.

One side of the proof follows directly from the definition.

Example 2.19: Consider the group ring Z_2S_3 of the symmetric group of degree 3 over the ring $Z_2 = \{0, 1\}$.
Take $S = \{0, 1 + p_1 + p_2 + p_3 + p_4 + p_5\}$ to be a subring of Z_2S_3. $P = \{0, 1 + p_1, 1 + p_2, 1 + p_3, 1 + p_4, 1 + p_5, p_4 + p_5\} \subseteq Z_2S_3$. Clearly P is a S-quasi set ideal of Z_2S_3 relative to the subring S. $S_2 = \{0, 1 + p_4, 1 + p_5, p_4 + p_5\} \subseteq Z_2S_3$ is a subring of Z_2S_3 and it is contained in P. But P is not a S-set ideal of Z_2S_3 as $S \not\subseteq P$.

Example 2.20: Let Z be the ring of integers. Take $S_1 = \{0, \pm 2, \pm 4, \ldots\}$, a subring of Z. $S_2 = \{0, \pm 3, \pm 6, \ldots\}$, subring of Z. Let $P = \{0, \pm 2, \pm 4, \ldots, \pm 3, \pm 9, \pm 15 \ldots\} \subseteq Z$.

P is a Smarandache set ideal of Z relative to S_1 as well as relative to S_2. Infact P is a S-quasi set ideal of Z relative to both S_1 and S_2. Also P is a S-strong quasi set ideal of Z.

Thus if P = $\{0, \pm 2, \pm 4, \ldots, \pm 3, \pm 6, \ldots, \pm 5, \pm 10, \ldots, \pm 7, \pm 14, \ldots, \pm 11, \pm 22, \ldots\} \subseteq Z$.

We see P is a Smarandache set ideal of Z relative to any one of the subrings;

$S_1 = \{0, \pm 2, \pm 4, \ldots\}$ or
$S_2 = \{0, \pm 3, \pm 6, \ldots\}$ or
$S_3 = \{0, \pm 5, \pm 10, \ldots\}$ or
$S_4 = \{0, \pm 7, \pm 14, \ldots\}$ or
$S_5 = \{0, \pm 11, \pm 22, \ldots\}$ or used not in the mutually exclusive sense.

In view of this we can say P is a S-strongly quasi set ideal of Z.

P is also a S-quasi set ideal of Z.

Now we proceed onto define the notion of Smarandache perfect set ideal ring.

DEFINITION 2.9: *Let R be any ring. Let $S_1, \ldots, S_n$, $S_i \not\subseteq S_j$; if $i \neq j$ ($n < \infty$) be the collection of all subrings of R.*

If $P = \{S_1 \cup \ldots \cup S_n\} \subseteq R$ is a proper subset of R and

(i) *P is a set ideal with respect to every subring S_i of R, $1 \leq i \leq n$.*
(ii) *P is a S-quasi set ideal of R with respect to every subring S_i of R, $1 \leq i \leq n$.*
(iii) *P is a S-strongly quasi set ideal of R with respect to every subring S_i of R; $1 \leq i \leq n$. Then we call R to be a Smarandache perfect set ideal ring.*

We illustrate this situation by some examples.

Example 2.21: Let Z_6 be the ring of integers modulo 6. The subrings of Z_6 are $S_1 = \{0, 3\}$ and $S_2 = \{0, 2, 4\}$.
$P = S_1 \cup S_2 = \{0, 3, 2, 4\} \subseteq Z_6$. R is a S-perfect set ideal ring.

Example 2.22: Let $Z_{10} = \{0, 1, 2, \ldots, 9\}$ be the ring of integers modulo 10. The subrings of Z_{10} as mentioned in the definition 2.10 are $S_1 = \{0, 5\}$ and $S_2 = \{0, 2, 4, 6, 8\}$.

$S_1 \cup S_2 = P = \{0, 5, 2, 4, 6, 8\} \subseteq Z_{10}$; P is a S-perfect set ideal ring.

In view of this we have the following theorem.

THEOREM 2.12: *Let $Z_{2p} = \{0, 1, 2, \ldots, 2p–1\}$ be the ring of integers modulo 2p, p a prime is a S-perfect set ideal ring.*

***Proof*:** The subrings of Z_{2p} mentioned in the above definition are $S_1 = \{0, p\}$ and $S_2 = \{0, 2, 4, \ldots, 2p–2\}$.

$P = S_1 \cup S_2$ is a S-perfect set ideal ring.

Example 2.23: Let $Z_{15} = \{0, 1, 2, \ldots, 14\}$ be the ring of integers modulo 15. The subrings of Z_{15} are $S_1 = \{0, 5, 10\}$ and $S_2 = \{0, 3, 6, 9, 12\}$. $P = S_1 \cup S_2 = \{0, 5, 10, 3, 6, 9, 12\} \subseteq Z_{15}$ is a S-quasi ideal of R. So Z_{15} is a S-perfect set ideal ring.

THEOREM 2.13: *Let $Z_n = \{0, 1, 2, \ldots, n–1\}$ where $n = pq$; $(p,q) = 1$, p and q primes. Z_n is a S-perfect set ideal ring.*

***Proof*:** (p, q) = 1, p and q are primes.

$S_1 = \{0, p, 2p, \ldots, p(q–1)\}$ and $S_2 = \{0, q, 2q, \ldots, (p–1)q\}$ are the only subrings of Z_n (n = pq).

$P = \{0, p, \ldots, p(q–1), q, \ldots, (p–1)q\} \underset{\neq}{\subset} Z_n$ and P is a S-strongly quasi set ideal Z_n. Thus Z_n is a S-perfect set ideal ring.

Example 2.24: Let $Z_{30} = \{0, 1, 2, \ldots, 29\}$ be the ring of integers modulo 30.

The subrings of Z_{30} as given by the definition 2.10 are $S_1 = \{0, 2, 4, 6, \ldots, 28\}$, $S_2 = \{0, 3, 6, \ldots, 27\}$ and $S_3 = \{0, 5, 10, \ldots, 25\}$. Clearly $S_i \not\subseteq S_j$ if $i \neq j$, $1 \leq i, j \leq 3$.

$P = S_1 \cup S_2 \cup S_3$

$= \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 3, 9, 15, 21, 27, 5, 25\} \subseteq Z_{30}$ is a S-strongly quasi set ideal of Z_{30} relative to the subring S_i, $i = 1, 2, 3$. Hence Z_{30} is a S- perfect set ideal ring.

Now we proceed onto define the notion of S-simple perfect set ideal rings.

DEFINITION 2.10: *Let R be a ring. If R has only one subring S and all other subrings are subrings of S then we call R to be a Smarandache simple perfect set ideal ring.*

We first illustrate it by examples.

Example 2.25: Let $Z_9 = \{0, 1, 2, \ldots, 8\}$ be the ring of integers modulo 9. $S = \{0, 3, 6\}$ is the only subring of Z_9. So Z_9 is a S-simple perfect set ideal ring.

Example 2.26: Let $Z_{16} = \{0, 1, 2, \ldots, 15\}$ be the ring of integers modulo 16. Take $S = \{0, 2, 4, 6, \ldots, 14\} \subseteq Z_{16}$. S is the only subring of Z_{16} and all other subrings of Z_{16} are subrings of S. Thus Z_{16} is also a S-simple perfect set ideal ring.

We show there exists a class of S-simple set ideal rings.

THEOREM 2.14: *Let Z_{p^n} be the ring of integers modulo p^n, p a prime, $n \geq 2$. Z_{p^n} is a S-simple set ideal ring.*

Proof: Given $Z_{p^n} = \{0, 1, 2, 3, \ldots, p, p+1, \ldots, p^n-1\}$ is the ring of integers modulo p^n, p a prime $n \geq 2$. The only subring as per definition 2.10 of Z_{p^n} is $S = \{0, p, 2p, \ldots, p^n-p\}$. All other subrings of Z_{p^n} are subrings of S. Thus Z_{p^n} is a S-simple set ideal ring.

Now Z_p when p is a prime has no proper subrings; that is why we assume in the theorem $n \geq 2$.

THEOREM 2.15: *Let $Z_n = \{0, 1, 2, \ldots, n-1\}$ where $n = p_1 p_2, \ldots, p_t$, where each p_i is a distinct prime. Then Z_n is a S-perfect set ideal ring.*

Proof: Given $Z_n = \{0, 1, 2, \ldots, n-1\}$ to be the ring of integers modulo n, where $n = p_1 p_2 \ldots p_t$. (p_i are primes, $p_i \neq p_j$, if $i \neq j$).

$S_1 = \{0, p_1, 2p_1, \ldots, (n-p_1)\}$,

$S_2 = \{0, p_2, 2p_2, \ldots, (n-p_2)\}, \ldots$

and $S_t = \{0, p_t, 2p_t, \ldots, (n-p_t)\}$ are the subrings of Z_n as given in definition.

Take $P = \{S_1 \cup \ldots \cup S_t\} \subseteq Z_n$ and P is a S-strong quasi set ideal of Z_n for every subring S_i, $i = 1, 2, \ldots, t$. Thus Z_n is a S-perfect set ideal ring.

We illustrate this by an example.

Example 2.27: Let $Z_n = \{0, 1, 2, \ldots, n-1\}$ where $n = 2.3.5.7.11.13$ be the ring of integers modulo n.

$S_1 = \{0, 2, \ldots, n-2\}$,

$S_2 = \{0, 3, \ldots, n-3\}$,

$S_3 = \{0, 5, \ldots, n-5\}$,

$S_4 = \{0, 7, \ldots, n-7\}$,

$S_5 = \{0, 11, \ldots, n-11\}$ and

$S_6 = \{0, 13, \ldots, n-13\}$ are subrings of Z_n.

Here $t = 6$.

$P = \{S_1 \cup S_2 \cup \ldots \cup S_6\} \subseteq Z_n$. Clearly P is a proper subset of Z_n and P is a S-strong quasi set ideal of Z_n. Hence Z_n is a S-perfect set ideal ring.

Now we proceed onto define the new notion of S-prime set ideals of a ring R which is commutative.

DEFINITION 2.11: *Let R be a commutative ring. P any non empty subset of R. Let $S \subseteq R$ be a subring of R which is a S-ring and not a field. We say P to be a Smarandache prime set ideal of R if the following conditions are true.*

(i) P is a S-set ideal of R relative to the S-subring S of R.
(ii) If $x.y \in P$ either x or y is in P.

We illustrate this situation by the following examples.

Example 2.28: Let $R = Z_{30} = \{0, 1, 2, \ldots, 29\}$ be the ring of integers modulo 29. Let $P = \{0, 2, \ldots, 26\}$ be a proper subset of R.

Take $S_1 = \{0, 10, 20\} \subseteq S = \{0, 5, 10, 15, 20, 25\} \subseteq R$. Clearly S is a S-subring of R as S_1 is a field isomorphic to Z_3. It can be easily verified; P is a S-prime set ideal of $R = Z_{30}$ relative to the S-subring S.

It is pertinent to mention here that in general all S-set ideals need not be S-prime set ideals, however trivially all S-prime set ideals are S-set ideals.

THEOREM 2.16: *Let R be a commutative ring. P a S-prime set ideal of R relative to the S-subring S of R. P is a S-set ideal of R. But in general a S-set ideal of a ring R need not be a S-prime set ideal of R relative to S.*

***Proof*:** By the very definition we know every S-prime set ideal of a ring R is a S-set ideal of R. To show in general a S-set ideal of a ring R need not be a S-prime set ideal of R, we give a counter example.

Take $Z_{12} = \{0, 1, 2, \ldots, 11\} = R$, the ring of integers modulo 12. Let $S = \{0, 2, 4, 6, 8, 10\} \subseteq Z_{12}$ be a S-subring of R. Let $P = \{0, 3, 4, 6, 8\} \subseteq R = Z_{12}$. P is a S-set ideal of R relative to

the subring S of R. But P is not a S-set prime ideal of R for 2.2 $\in$ P but 2 $\notin$ P; 2.3 $\in$ P but 2 and 3 $\notin$ P.

Hence the claim.

Consider the following example.

Example 2.29: Let R = Z_4 = {0, 1, 2, 3} be the ring of integers modulo 4. Let S = {0, 2} be the subring of R = Z_4.

P = {0, 1, 2} $\subseteq$ Z_4 is a set ideal of R relative to subring S of R. We see 3.3 = 1 $\in$ P but 3 $\notin$ P. Also if we take P_1 = {0, 3, 2} $\subseteq$ Z_4, P_1 is a set ideal of Z_4 relative to the subring S = {0, 2}.

In view of this we define first the notion of prime set ideals of a ring R.

DEFINITION 2.12: *Let R be a commutative ring. S a subring of R. P a proper subset of R such that P is a set ideal of R relative to the subring S. If for every x.y $\in$ P either x or y is in P then we call P to be a prime set ideal of R relative to the subring S of R.*

We first illustrate this situation by some examples.

Example 2.30: Let Z_{14} = {0, 1, 2, …, 13} be the ring of integers modulo 14. let S = {0, 7} $\subseteq$ Z_{14} be a subring of R. Take P = {0, 2, 4, 8, 10} $\subseteq$ Z_{14}. P is a prime set ideal of Z_{14} relative to the subring, S = {0, 7}.

Take P_1 = {0, 1, 7, 2, 4} $\subseteq$ Z_{14}. P_1 is only a set ideal of Z_{14} relative to the subring S = {0, 7} of Z_{14}. For 13.13 = 1 $\in$ P_1 but 13 $\notin$ P_1 so P_1 is not a set prime ideal or prime set ideal of R = Z_{14}.

In view of this we have the following theorem.

THEOREM 2.17: *Let R be a commutative ring. Every prime set ideal P of R relative to a subring S of R is a set ideal of R.*

However in general a set ideal of R need not be a prime set ideal of R.

The proof of the following theorem is left as an exercise for the reader.

Now we observe the following interesting property from the example given below.

Example 2.31: Let R = Z_{21} = {0, 1, 2, …, 20} be the ring of integers modulo 21. S ={0, 7, 14} is a subring of Z_{21}. Now P = {0, 3, 6, 9, 12} $\subseteq$ Z_{21} is a set ideal of R which is also a prime set ideal of R relative to the subring S. We see P is not a S-prime set ideal of R as P is not a even a S-set ideal of R.

We see that a prime set ideal in the first place need not be even a S-set ideal. Secondly in general a prime set ideal need not be a S-prime set ideal.

In view of this we leave the following theorems as exercise to the reader.

THEOREM 2.18: *A set prime ideal of a ring R in general need not be a S-set ideal of a ring R.*

THEOREM 2.19: *A set prime ideal of a ring R in general need not be a S-set prime ideal of R.*

Now we propose the following interesting problems to the reader.

Problem: Find conditions on the ring R, so that the subring of R is a prime set ideal and is also a S-prime set ideal of R.

Problem: Find conditions on the ring R and on the subrings S of R so that a prime set ideal of a ring R is a S-set ideal of the ring R but not a S-prime set ideal of R.

We prove the following interesting result about the ring of integers Z_n where n = pq; p and q are distinct primes.

THEOREM 2.20: *Let $R = Z_n = \{0, 1, 2, \ldots, n-1\}$ be the ring of integers modulo n where n = pq where p and q are distinct primes. Any prime set ideal of Z_n will never be a S-set ideal of the ring Z_n relative to a subring S of Z_n.*

***Proof*:** Given $R = Z_n = \{0, 1, 2, \ldots, n-1\}$ is the ring of integers modulo n where n = pq, with p and q distinct prime. This ensures Z_n has only two subrings $S_1 = \langle p \rangle$ and $S_2 = \langle q \rangle$ where S_1 and S_2 are themselves prime fields of characteristic p and q respectively i.e., $S_1 \cong Z_p$ and $S_2 \cong Z_q$. So Z_n has no S-subrings, this in turn forces any set ideal P of Z_n to be non S-set ideal of Z_n. Thus even if P is a prime set ideal of $R = Z_n$ relative to S_1 or S_2, P can never be a S-set ideal of Z_n, hence cannot be a S-prime set ideal of Z_n.

We illustrate this situation by the following examples.

Example 2.32: Let $Z_{21} = \{0, 1, 2, \ldots, 20\}$ be the ring of integers modulo 21. Here n = 21 = pq = 7.3. The only subrings of Z_{21} are $S_1 = \{0, 3, 6, 9, 12, 15, 18\} \cong Z_7$ and $S_2 = \{0, 7, 14\} \cong Z_3$. Thus any prime set ideal of Z_{21} relative to the subrings S_1 or S_2 cannot be even S-set ideals of Z_{21} as S_1 and S_2 are not S-subrings of Z_{21}. Infact Z_{21} has no S-subrings.

Hence the claim.

Example 2.33: Let $Z_{26} = \{0, 1, 2, \ldots, 25\}$ be the ring of integers modulo 26. Here n = 26 = 2.13 = p.q. The only subrings of Z_{26} are $S_1 = \{0, 13\} \cong Z_2$ and $S_2 = \{0, 2, 4, 6, 8, \ldots, 24\} \cong Z_{13}$. Thus Z_{26} has no S-subrings so Z_{26} cannot have S-set prime ideals or S-set ideals as Z_{26} has no S-subrings.

We propose an interesting problem.

Problem: Characterize all rings which has

(1) S-prime set ideals.

(2) S-set ideals.

Problem: Does there exist other rings other than Z_{pq} (p and q distinct primes) which has no S-subrings.

Now we proceed onto prove yet another interesting result which guarantees the existence S-set prime ideals.

THEOREM 2.21: *Let Z_n = {0, 1, 2, …, n–1} be the ring of integers modulo n where n = p q r where p, q and r three distinct primes. Then Z_n has nontrivial S-set ideals P which are S-prime set ideals provided P is also a prime set ideal of Z_n.*

Proof: Take S_1 to be a S-subring of order pq or pr or qr. Clearly any prime set ideal of Z_n relative to S_1 will also be a S-set prime ideal of Z_n.

We can extend this result to any ring of modulo integers n where n = $p_1 \ldots p_t$, $p_1, \ldots, p_t$ are distinct primes or n = $p_1^{\alpha_1} \ldots p_s^{\alpha_s}$ where $p_1, \ldots, p_s$ are distinct primes $\alpha_i \geq 1$; $1 \leq i \leq s$.

We illustrate this situation by the following examples.

Example 2.34: Let Z_{30} = {0, 1, 2, …, 29} be the ring of integers modulo 30. Here n = 30 = 2.3.5 = pqr.

Take S_1 = {0, 2, 4, …, 28} $\cong Z_{15}$. S_1 is a S-subring of Z_{30}. Let P = {0, 2, 4, 6, …, 28, 5} $\subseteq Z_{30}$, P is S-prime set ideal of Z_{30} relative to the S-subring S_1.

Similarly S_2 = {0, 5, 10, 15, 20, 25} $\subseteq Z_{30}$ is a S-subring of Z_{30} which is isomorphic to Z_6. Take P = {0, 2, 20, 10, 4} $\subseteq Z_{30}$, P is a S-prime set ideal of Z_{30} relative to the S-subring S_2.

If S_3 = {0, 3, 6, 9, 12, 15, 21, 24, 27} $\subseteq Z_{30}$, S_3 is a S-subring of Z_{30} isomorphic to Z_{10}.

It is left as an exercise for the reader to find a S-prime set ideal of Z_{30} relative to S_3.

Example 2.35: Let $Z_{18} = \{0, 1, 2, \ldots, 17\}$ be the ring of integers modulo 18.

Take $S = \{0, 2, 4, 6, \ldots, 16\}$ a S-subring of Z_{18} of order 9. It is easily verified that one can find a subset P of Z_{18} which is a S-set prime ideal of Z_{18} relative to S.

Does their exists infinite commutative rings which has no S-prime set ideals?

Can Z have S-prime set ideals?
Does C have S-prime set ideal?

The above three problems are left as an exercise for the reader.
Thus we have seen there exists non trivial class of S-perfect set ideal rings.

Problems:

1. Find a S-set ideal in Z_{35}.
2. Can Z_{64} have a S-strong quasi set ideal?
3. Can Z_{3^5} be a S-perfect set ideal ring? Justify your claim.
4. Find for the group ring ZS_3.

   (i) Set ideal

   (ii) S-set ideal

   (iii) S-quasi set ideal.
5. Obtain some interesting properties about S-perfect set ideal ring.
6. If R is a S-perfect set ideal ring, will RG be a S-perfect set ideal ring for any group G?

7. Suppose R is a S-perfect set ideal ring R.

   Will RS the semigroup ring of the semigroup S over the ring R be a S-perfect set ideal ring for any semigroup S?
8. What can one say about the semigroup ring $Z_{15}S(3)$?
9. Find semigroup rings RG which are not S-perfect set ideal rings.
10. Give examples of groupring RG which are S-perfect set ideal rings?
11. Will Z_{11} S_7 be a S-perfect set ideal ring?
12. Can $Z_{12}S_4$ be a S-perfect set ideal ring?
13. Can Z_9S_3 be a S-perfect set ideal ring? Justify your claim.
14. Will ZS_7 have S- set ideals?
15. Can ZD_8 have S-quasi set ideal?
16. Can ZA_5 have S-strong quasi set ideals?
17. Find set ideals of the group ring ZG where $G = \langle g \mid g^{12} = 1\rangle$.
18. Can the group ring ZG where G is an infinite cyclic group have S-strong quasi set ideal?
19. Does their exists groups G for which the group ring Z_5G can have S-set ideals?
20. Give an example of a group ring which is S-simple perfect set ideal ring.
21. Does their exist a semigroup ring RS which is a S-simple perfect set ideal ring?
22. Can the group ring Z_{81} S_3 be a S-simple perfect set ideal groupring?

23. Will Z_7S_3 be a S-simple perfect ideal group ring? Justify your claim!

24. Can $Z_{11}G$ where $G = \langle g \mid g^7 = 1\rangle$, the group ring of the group G over the ring Z_{11} be a S-simple perfect ideal group ring?

25. If R is a S-simple perfect ideal ring for any group G over the ring R, can the group ring RG be a S-perfect set ideal ring?

26. Give examples of group rings which are not S-perfect simple set ideal rings.

27. Give examples of group rings which are not S-perfect set ideal rings.

28. Obtain some interesting properties about S-simple perfect set ideal rings.

Chapter Three

SET IDEAL TOPOLOGICAL SPACES

In this chapter we proceed onto define set ideals in semigroups, S-set ideals in semigroups, S-quasi set ideals in semigroups and finally using the collection of set ideals of a semigroup S relative to a subsemigroup S_1, we construct set ideal topological space of a semigroup relative to the subsemigroup. We also give the lattice representation for this collection.

Throughout this chapter S will denote a semigroup commutative or otherwise.

DEFINITION 3.1: *Let S be a semigroup, S_1 a proper subsemigroup of S. Let $P \subset S$ where P is just a proper subset of S. If for every $p \in P$ and for every $s \in S_1$, sp and ps are in P then we call P to be a set ideal of S relative to or over to the subsemigroup S_1 of S.*

We illustrate this by some examples.

Example 3.1: Let Z_{18} be the semigroup under multiplication modulo 18. $S_1 = \{0\ 3,\ 9\}$ is a subsemigroup of Z_{18}. $P = \{0, 4, 6, 8, 12\} \subseteq Z_{18}$ is a set ideal of Z_{18} relative to the subsemigroup $S_1 = \{0, 3, 9\}$.

Example 3.2: Let $Z_{15} = \{0, 1, 2, \ldots, 14\}$ be the semigroup under multiplication modulo 15. Let $S_1 = \{0, 5, 10\}$ be the subsemigroup of Z_{15}. Take $P = \{0, 3, 6, 12\} \subseteq Z_{15}$ to be a proper subset of Z_{15}. P is a set ideal of Z_{15} relative to the subsemigroup $S_1 = \{0, 5, 10\}$ of S.

Every semigroup contains a nontrivial set ideal relative to some subsemigroup. Is this true?

Example 3.3: Take $S = \{1, -1\}$ a semigroup under multiplication. $P = \{-1\} \subset S$ and take the subsemigroup $S_1 = \{1\}$. Clearly P is a ideal of S relative to S_1, the subsemigroup of S.

Example 3.4: Consider the semigroup $Z_3 = \{0, 1, 2\}$ under multiplication modulo 3. $S_1 = \{1, 2\}$ is a subsemigroup of Z_3. $P = \{0\} \subset Z_3$ is the set ideal of Z_3 relative to the subsemigroup S of Z_3.

Example 3.5: Let $Z_5 = \{0, 1, 2, 3, 4\}$ be the semigroup under multiplication modulo 5. Take $S_1 = \{1, 4\}$ the subsemigroup of Z_5. Take $P = \{2, 3\} \subseteq Z_5$. P is the set ideal of Z_5.

However it is left for the reader to show whether Z_p has a set ideals or not.

Clearly Z_{11} has the set ideal other than the set $P = \{0\}$ is given by $P = \{0, 2, 3, 4, 5, 6, 7, 8, 9\}$. P is easily verified to be a set ideal of Z_{11} relative to the subsemigroups $\{0, 1\}$, $\{1, 11\}$ and $\{0, 1, 11\}$.

Thus we have the following theorem.

THEOREM 3.1: *Let $Z_p = \{0, 1, 2, \ldots, p-1\}$ be the semigroup under multiplication modulo p has set ideals. $S = \{1, p-1\}$ is a subsemigroup of S.*

Proof: Take $P_1 = \{0\}$, P_1 is a set ideal of Z_p relative to the semigroup S.

Consider $P_2 = \{0, 2, 3, 4, \ldots, p-2\} = Z_p \setminus \{1, p-1\} \subseteq Z_p$, P is a set ideal of Z_p relative to the subsemigroup $S = \{0, 1, p-1\}$ and the subsemigroup $S_1 = \{p-1, 1\}$.

Now we proceed onto define the notion of S-set ideals of a semigroup.

DEFINITION 3.2: *Let S be a semigroup. S_1 a subsemigroup of S. $P \subset S$ be a proper subset. Suppose P is a set ideal of S relative to the subsemigroup S_1 and if $S_1 \subseteq P$ then we call P to be a Smarandache set ideal (S-set ideal) of the semigroup S relative to the subsemigroup S_1.*

Example 3.6: Let $S = Z_{20}$ be the semigroup under multiplication modulo 20. $S_1 = \{0, 10\} \subseteq Z_{20}$ is a subsemigroup of Z_{20}. Take $P = \{0, 4, 6, 5, 9, 10\} \subseteq Z_{20}$ is a S-set ideal of Z_{20} relative to the subsemigroup S_1.

Example 3.7: Let $Z_{24} = S$ be the semigroup under multiplication modulo 24. $S_1 = \{0, 12\}$ be the subsemigroup of S. Take $P = \{0, 2, 6, 16, 18, 20, 12\} \subseteq Z_{24}$. P is a S-set ideal of Z_{24} relative to the subsemigroup S_1 of $S = Z_{24}$.

Example 3.8: Let $Z_{19} = \{0, 1, 2, \ldots, 18\}$ be the semigroup under multiplication modulo 19. $S = \{0, 1, 18\}$ is a subsemigroup of Z_{19}. Take $P = \{0, 1, 18, 2, 17\} \subseteq Z_{19}$, P is a S-set ideal of Z_{19}.

Now we show all set ideal semigroup are not S-set ideal semigroup, but all Smarandache set ideal semigroups are set ideal semigroups. To this end we give an example.

Example 3.9: Let $Z_{21} = \{0, 1, 2, \ldots, 20\}$ be the semigroup under multiplication modulo 21. Take $S = \{0, 7\}$ to be the subsemigroup of Z_{21}. Let $P = \{0, 3, 6, 12, 15\} \subseteq Z_{21}$. P is a set

ideal of the semigroup Z_{21}, but P is not a S-set ideal of Z_{21} relative to the subsemigroup S.

Now by the very definition of Smarandache set ideal of a semigroup it is also a set ideal. We proceed onto define the notion of Smarandache quasi set ideal of a semigroup with respect to a subsemigroup of the semigroup.

DEFINITION 3.3: *Let S be a semigroup. S_1 a subsemigroup of S. Let $P \subseteq S$ be a set ideal of S relative to the subsemigroup S_1 of S. If P contains a subsemigroup $S_2 \neq S_1$ of S then we call P to be Smarandache quasi set ideal of S relative to the subsemigroup S_1 of S.*

We illustrate this situation by some examples.

Example 3.10: Let $Z_{24} = \{0, 1, 2, \ldots, 23\}$ be the semigroup under multiplication modulo 24. Take $S_1 = \{0, 12\}$ a subsemigroup of Z_{24}. Let $P = \{0, 8, 16, 2, 4, 10\} \subseteq Z_{24}$ is a S-quasi set ideal of Z_{24} relative to the subsemigroup $S_1 = \{0, 12\}$ as $T = \{0, 4, 16, 8\} \subseteq Z_{24}$ is a subsemigroup of Z_{24}.

Example 3.11: Let $Z_{25} = \{0, 1, 2, 3, \ldots, 24\}$ be the semigroup under multiplication modulo 25. Let $S = \{0, 5\}$ be the subsemigroup of Z_{25}. Take $P = \{0, 10, 15, 20\} \subseteq Z_{25}$. P is a S-quasi set ideal of Z_{25} relative to the semigroup S. P contains subsemigroups like $S_2 = \{0, 10\}$, $S_3 = \{0, 20\}$, $S_4 = \{0, 10, 20\}$ and $S_5 = \{0, 5, 10\}$ of Z_{25}.

A S-quasi set ideal in general is not a S-set ideal of the semigroup.

Further every S-set ideal in general need not be a S-quasi set ideal. We prove these assertions only by examples.

Example 3.12: Let $Z_{12} = \{0, 1, 2, \ldots, 11\}$ be the semigroup under multiplication modulo 12. Let $S = \{0, 6\}$ be the subsemigroup of Z_{12}. Take $P = \{0, 2, 4\} \subseteq Z_{12}$, P is a S-quasi set ideal of Z_{12} relative to the subsemigroup S of Z_{12}.

For $S_2 = \{0, 4\}$ is a subsemigroup of Z_{12} but $S \not\subseteq P$ so P is not a S-set ideal of Z_{12} relative to S.

To show a S set ideal of a semigroup S relative to a subsemigroup S_1 need not be a S-quasi set ideal. We construct the following example.

Example 3.13: Let $Z_{28} = \{0, 1, 2, \ldots, 27\}$ be the semigroup under multiplication modulo 28. Take $S = \{1, 27\}$ to a subsemigroup of Z_{28}. $P = \{0, 2, 26, 3, 25\} \subseteq Z_{28}$ is not a S-quasi set ideal as P does not contain any subsemigroup.

Now consider $S_1 = \{0, 7, 21\} \subseteq Z_{28}$, S_1 is a subsemigroup of Z_{28}. Take $P_1 = \{0, 4, 8, 16\} \subseteq Z_{28}$; P_1 is a S-quasi set ideal of Z_{28} relative to the subsemigroup S_1. $S_2 = \{4, 8, 16\} \subseteq P_1$ is a subsemigroup of P_1. But $S_1 \not\subseteq P_1$ so P_1 is not a S set ideal of Z_{28} relative to the subsemigroup S_1.

Now we proceed onto define Smarandache perfect quasi set ideal of a semigroup S.

DEFINITION 3.4: *Let S be a semigroup. Let $S_1, \ldots, S_t$ be the collection of all subsemigroups of S such that $S_i \not\subseteq S_j$; if $i \neq j$, $1 \leq i, j \leq n$. Let $P = S_1 \cup \ldots \cup S_t$, if $P \subseteq S$ we say P is a Smaradache perfect quasi set ideal semigroup if, P is a Smarandache set ideal of S relative to every subsemigroup S_i of S; $i = 1, 2, \ldots, t$.*

We illustrate this by the following examples.

Example 3.14: Let $Z_6 = \{0, 1, 2, \ldots, 5\}$ be the semigroup under multiplication modulo 6. The subsemigroup of Z_6 satisfying the conditions of the definition are $S_1 = \{0, 3, 1\}$, $S_2 = \{0, 2, 4, 1\}$ and $S_3 = \{0, 1, 5\}$, $S_1 \cup S_2 \cup S_3 = P = Z_6$ so Z_6 is trivially a S-perfect quasi set ideal semigroup.

Example 3.15: Let $S = \{0, 1, 2, 3, 4\} = Z_5$ semigroup under multiplication modulo 5. The subsemigroups of S are $S_1 = \{1, 4\}$, $S_2 = \{0, 1, 4\}$. So we can take only S_2 as $S_1 \subset S_2$.

Since S has only one subsemigroup as given in the definition 3.4; S is not a S-perfect quasi set ideal semigroup.

Example 3.16: Consider the semigroup $Z_{10} = \{0, 1, 2, \ldots, 9\}$ under multiplication modulo 10. The subsemigroups of Z_{10} as given in the definition are $S_1 = \{0, 7, 9, 1, 3\}$, $S_2 = \{0, 5\}$ and $\{0, 2, 4, 6, 8\} = S_3$.

Now $S_1 \cup S_2 \cup S_3 \supseteq Z_{10}$ i.e., $Z_{10} = S_1 \cup S_2 \cup S_3$, so in this case also Z_{10} is trivially a S-perfect quasi set ideal semigroup.

Example 3.17: Let $Z_{30} = \{0, 1, 2, \ldots, 29\}$ be the semigroup under multiplication modulo 30.

The subsemigroups of Z_{30} are;
$S_1 = \{0, 1, 29\}$, $S_2 = \{2, 4, 8, 16\}$, $S_3 = \{0, 10, 20\}$,
$S_4 = \{0, 14, 16, 22, 18\}$, $S_5 = \{0, 9, 17, 3, 27, 21\}$,
$S_6 = \{0, 15\}$, $S_7 = \{1, 17, 19, 23\}$, $S_8 = \{0, 5, 25\}$,
$S_9 = \{0, 24, 12, 18, 6\}$, $S_{10} = \{0, 16, 26\}$,
$S_{11} = \{0, 28, 4, 16, 22\}$, $S_{12} = \{0, 7, 19, 13, 1\}$
and $S_{13} = \{0, 11\}$, $Z_{30} \subseteq \bigcup_{i=1}^{13} S_i$.

Clearly Z_{30} is a S-perfect quasi set ideal semigroup.

It is pertinent to mention here that for a semigroup S which is S-perfect quasi set ideal semigroup we have

$$S \subset \bigcup_{i=1}^{n} S_i \ (n < \infty) \text{ and } \bigcup_{i=1}^{n} S_i \subseteq S \text{ so } S = \bigcup_{i=1}^{n} S_i .$$

Thus it is very rare to find $\bigcup_{i=1}^{n} S_i \underset{\neq}{\subset} S$ (strict inequality or containment $n \neq 1$, that is $n > 1$).

We have given only examples from the semigroups Z_n; n any integer.

We find the following interesting results about set ideals in semigroup or set ideal semigroups.

Example 3.18: Let Z_{12} = {0, 1, 2, …, 11} be a semigroup under ×. The subsemigroups of Z_{12} are S_1 = {0, 6, 1}, S_2 = {0, 6}, S_3 = {0, 3, 6, 9}, S_4 = {1, 0, 9, 6, 3}, S_5 = {0, 4, 8, 1}, S_6 = {0, 4, 8}, S_7 = {0, 1} and S_8 = {0, 11, 1}.

Consider the subsemigroup S_6 = {0, 4, 8} $\subseteq Z_{12}$.

Let P = {0, 3, 6, 9} $\subseteq Z_{12}$. P is a set ideal of the semigroup relative to the subsemigroup S_6.

Clearly P is also a set ideal of the semigroup relative to the subsemigroup S_7.

We see P is also a set ideal of the semigroup Z_{12} relative to the subsemigroup $S_6 \cap S_7$ = {0} as well as $S_6 \cup S_7$ = {0, 1, 4, 8} = S_5.

In such cases we define the set ideal to be a nice set ideal. In case $S_i \cup S_j$ is not a semigroup or the full semigroup. We call such set ideals as bad set ideals.

Example 3.19: Let Z_7 be the semigroup under multiplication modulo 7. S_1 = {0, 1, 6} $\subseteq Z_7$ is a subsemigroup of Z_7. S_2 = {0, 1, 2, 4} $\subseteq Z_7$ is a subsemigroup of Z_7. S_3 = {0, 1, 5, 4, 6, 2, 3} = Z_7 is a improper subsemigroup of Z_7.

P = {0, 5, 2} $\subseteq Z_7$ is a set ideal of Z_7 relative to the subsemigroup {0, 6} = S. However P is not a set ideal of Z_7 relative to the subsemigroup S_2 = {0, 1, 2, 4}.

P_1 = {0, 3, 5, 6} $\subseteq Z_7$ is a set ideal of Z_7 relative to the subsemigroup S_2 = {0, 1, 2, 4} (or S_3 = {0, 4, 2}).

P_1 is not a set ideal over the subsemigroup S_1 = {0, 1, 6}. For S_1P_1 = {0, 4, 2, 1}. Thus $S_1P_1 \not\subseteq P_1$, hence P_1 is not a set ideal relative to the subsemigroup S_1.

In view of this example we make the following definition.

DEFINITION 3.5: *Let S be a semigroup. P a proper subset of S, S_1 and S_2 be two distinct subsemigroups of S. If P is a set ideal of S with respect to the semigroup S_1 (say) and not a set ideal of S with respect to S_2 but $PS_2 = S_1$ then we call the subsemigroup S_2 to be a set ideal related to subsemigroup S_1 via P.*

However S_1 is not a set ideal related subsemigroup of S_2 via P.

The above example is an illustration of this definition.

Example 3.20: Let Z_{11} be the semigroup under product. The subsemigroup of Z_{11} are $S_1 = \{0, 1\}$, $S_2 = \{0, 1, 10\}$, $S_3 = \{1, 3, 9, 4, 5\}$ and $S_4 = \{0, 1, 3, 4, 5, 9\}$ are some of the subsemigroups of Z_{11} under modulo 11. We now find set ideals of Z_{11} related to some of these subsemigroups.

Take $P = \{0, 7, 4\} \subseteq Z_{11}$ is a set ideal of the semigroup Z_{11} over the subsemigroup $S_2 = \{0, 1, 10\} \subseteq Z_{11}$.

However $P \subseteq Z_{11}$ is not a set ideal of the semigroup Z_{11} over the subsemigroup $S_3 = \{1, 3, 9, 4, 5\}$ or S_4; but is a set ideal over S_1.

It is interesting to see that Z_p, p a prime has non trivial subsemigroups under $\times$. Further Z_p has set ideals, but Z_p does not contain any ideals other than $\{0\}$.

This is one of the advantages of using set ideals and the major difference between the set ideals and ideals of a semigroup of modulo integers for a prime p.

Example 3.21: Let $S = Z_{13} = \{0, 1, 2, \ldots, 12\}$ be the semigroup under product modulo 13.

Clearly $P_1 = \{0, 1, 12\} \subseteq Z_{13}$ is a subsemigroup of the semigroup Z_{13}. $P_2 = \{1, 12\} \subseteq Z_{13}$ is also a subsemigroup of Z_{13}. 2 generates $Z_{13} \setminus \{0\}$ so only $P_3 = Z_{13} \setminus \{0\} \subseteq Z_{13}$ can be semigroup which is also a group. P_2 is also a group.

$M_1 = \{0, 2, 11\} \subseteq Z_{13}$ is a set ideal of Z_{13} over the subsemigroups P_1 and P_2.

$M_2 = \{0, 3, 10\} \subseteq Z_{13}$ is also a set ideal of Z_{13} over the subsemigroups P_1 and P_2. $M_3 = \{0, 4, 9\} \subseteq Z_{13}$ is also a set ideal of Z_{13} over the subsemigroups P_1 and P_2 of Z_{13}.

$M_4 = \{0, 5, 8\} \subseteq Z_{13}$ is also a set ideal of Z_{13} over the subsemigroups P_1 and P_2. $M_5 = \{0, 6, 7\} \subseteq Z_{13}$ is also a set ideal of Z_{13} over the subsemigroups P_1 and P_2.

Now based on this example we make the following definition.

DEFINITION 3.6: *Let S be a semigroup. $P \subseteq S$ be a proper subset of S. $G \subseteq S$ be a group in S that is S is a Smarandache semigroup. If P is a set ideal over G we call P to be a strong set ideal of the semigroup S over the group G of S.*

We will first give examples of them.

Example 3.22: Let $S = Z_{17}$ be the semigroup under product $\times$. $P = \{0, 1, 16\}$ is a subsemigroup of S.

Consider $M_1 = \{0, 2, 15\} \subseteq Z_{17}$ is a strong set ideal of S over the subsemigroup $P = \{0, 1, 16\}$.

Likewise $M_2 = \{0, 3, 14\}$, $M_3 = \{0, 4, 13\}$, $M_4 = \{0, 5, 12\}$, $M_5 = \{6, 0, 11\}$, $M_6 = \{0, 7, 10\}$ and $M_7 = \{0, 8, 9\}$ are set ideals of the semigroup S over the subsemigroup $P = \{0, 1, 16\}$. If in P we remove '0' and call it as $P_1 = \{1, 16\}$ then $M_2, \ldots, M_8$ are strong set ideals of the semigroup over the group P_1. Also $M_i \setminus \{0\}$; $1 \leq i \leq 7$ are strong set ideals of the semigroup over the group $P_1 = \{1, 16\}$. Thus Z_{17} has atleast 14 such strong set ideals over the group $P_1 = \{1, 16\}$.

In view of this we have the following theorem.

THEOREM 3.2: *Let Z_p = {0, 1, 2, …, p–1}, p a prime be the semigroup under $\times$ mod p. Let P = {1, p–1} be the group of Z_p. Z_p has (p–3) strong set ideals relative to P.*

***Proof*:** Take $M_1 = \{2, p-2\}$, $M_2 = \{3, p-3\}$, $M_3 = \{4, p-4\}$,… and $M_{(p-3)/2} = \left\{\frac{p-1}{2}, \frac{p+1}{2}\right\}$ to be strong set ideals of Z_p over P = {1, p–1}.

Also $N_1 = \{0, 2, p-2\}$, $N_2 = \{0, 3, p-3\}$, …, and $M_{(p-3)/2} = \left\{0, \frac{p-1}{2}, \frac{p+1}{2}\right\}$ are strong set ideals of Z_p over P. Thus Z_p has p–3 such strong set ideals over P.

Corollary 3.1: Every one of the p–3 strong set ideals of Z_p over Z_p are such that the sum of its non zero terms is p.

Proof: Obvious from the fact for any M_j or N_j in theorem 3.2 we see $a + b \equiv p$ for $a, b \in M_j$ or $a, b \in N_i \setminus \{0\}$; for $1 \leq i, j \leq \left(\frac{p-3}{2}\right)$.

The reader can study whether the theorem 3.2 holds good in case of Z_n, n not a prime.

In view of this we consider the following examples.

Example 3.23: Let Z_{15} be semigroup under product $\times$. $S = \{1, 14\} \subseteq Z_{15}$ is a group under product.

Consider $P_1 = \{0, 2, 13\}$, $P_2 = \{0, 3, 12\}$, $P_3 = \{0, 4, 11\}$, $P_4 = \{0, 5, 10\}$, $P_5 = \{0, 6, 9\}$ and $P_6 = \{0, 7, 8\}$ are strong set ideals of Z_{15} over the group S.

Also $M_1 = \{2, 13\}$, $M_2 = \{3, 12\}$, $M_3 = \{4, 11\}$, $M_4 = \{5, 10\}$, $M_5 = \{6, 9\}$ and $M_6 = \{7, 8\}$ are strong set ideals of Z_{15} over S.

Thus Z_{15} has 12 strong set ideals over the group $S \subseteq Z_{15}$.

Example 3.24: Let Z_{32} be the semigroup under product. $P = \{1, 31\}$ be the group under product. Every $M_i = \{0, t, 32–t\}$ and $N_j = \{t, 32–t\}$, $t \geq 2$, $2 \leq t \leq 15$. $1 \leq i, j \leq 14$ are strong set ideals of Z_{32} over the group P and $\{0, 16\} \subseteq Z_{32}$ is also a strong set ideal of Z_{32} over P.

Example 3.25: Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the semigroup under product. $P = \{1, 5\} \subseteq Z_6$ is a group. $M_1 = \{2, 4\}$ and $M_2 = \{3\}$ are strong set ideals of Z_6 over P.

Further $N_1 = \{0, 2, 4\}$ and $N_2 = \{0, 3\}$ are also set ideals of Z_6 over the group $\{1, 5\} \subseteq Z_6$.

Example 3.26: Let $S = Z_{18}$ be the semigroup under product. $P = \{1, 17\} \subseteq S$ be the group under product.

$M_1 = \{2, 16\}$, $M_2 = \{3, 15\}$, $M_3 = \{4, 14\}$, $M_4 = \{5, 13\}$, $M_5 = \{6, 12\}$, $M_6 = \{7, 11\}$, $M_7 = \{8, 10\}$ and $M_8 = \{9\}$ are subsets of S which are strong set ideals of S over the group $P \subseteq S$.

Also $N_1 = \{0, 2, 16\}$, $N_2 = \{0, 3, 15\}$, …, $N_8 = \{0, 9\}$, that is $N_j = M_j \cup \{0\}$; $1 \leq j \leq 8$ are also subsets of S which are strong set ideals of S over the group $P = \{1, 17\}$.

In view of all this we have the following theorem.

THEOREM 3.3: *Let Z_n be the semigroup under product (n any composite number of the from 2nz) Z_n has atleast n – 2 number of strong set ideals over a group $G \subseteq Z_n$.*

Proof: Take $G = \{1, n–1\} \subseteq Z_n$, G is a group under product. Let $M_j = \{t, n–t\}$ and $N_j = M_j \cup \{0\}$. $1 \leq j \leq (n–2)/2$ and $2 \leq t \leq n/2$ or $n–1/2$ M_j and N_j are strong set ideal of Z_n over the group G and they are n–2 in number.

Our natural question would be can Z_n have more strong set ideals of the semigroup over any other group.

To this end we make the following observations.

Example 3.27: Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the semigroup under product. Apart from the group $G = \{1, 5\} \subseteq Z_6$, $H = \{2, 4\} \subseteq Z_6$ is a again group under product and 4 act as the identity. The table of H is as follows;

×	2	4
2	4	2
4	2	4

.

Now $P = \{0\} \subseteq Z_6$ is a strong set ideal of Z_6 over the group $H = \{2, 4\}$. Also P is a strong set ideal of Z_6 over the group $G = \{1, 5\}$. However $H \cap G = \phi$. We see P is set ideal over the H is a set ideal over the semigroup P of Z_6 as $H \subseteq Z_6$; H is a semigroup under product. $M = \{0, 3, 5, 2, 4\} \subseteq Z_6$ is a strong set ideal over the group $G = \{1, 5\}$ and $H = \{2, 4\}$. But it is to be noted that $H \subseteq M$.

Example 3.28: Let Z_{12} be the semigroup under ×.

$G_1 = \{4, 8\} \subseteq Z_{12}$ is a group with 4 as its identity; $8^2 \equiv 4 \pmod{12}$, $G_2 = \{3, 9\} \subseteq Z_{12}$ is again a group with 9 as its identity $3^2 \equiv 9$ and $9^2 \equiv 9 \pmod{12}$, $G_3 = \{1, 5\} \subseteq Z_{12}$ is a group for $5^2 = 1 \pmod{12}$, $G_4 = \{1, 7\} \subseteq Z_{12}$ is a group for $7^2 \equiv 1 \pmod{12}$ and 1 is the identity. $G_5 = \{1, 11\} \subseteq Z_{12}$ is again a group of Z_{12}.

Let $P = \{2, 10\} \subseteq Z_{12}$ be a set. Clearly P is not closed under product. P is a strong set ideal of Z_{12} over the group $G_5 = \{1, 11\}$. However P is not a strong set of Z_{12} over the group G_1 or G_2. However P is a strong set ideal of Z_{12} over the group $G_4 = \{1, 7\} \subseteq Z_{12}$ and $G_3 = \{1, 5\} \subseteq Z_{12}$. Thus P is a strong set ideal of Z_{12} over the three groups $G_4 = \{1, 7\}$, $G_3 = \{1, 5\}$ and

$G_5 = \{1, 11\}$ of Z_{12}. Consider $T = \{2, 6\} \subseteq Z_{12}$; T is a strong set ideal over the groups $G_5 = \{1, 11\}$, $G_4 = \{1, 7\}$ and $G_2 = \{3, 9\}$ of Z_{12}.

Suppose $B = \{2, 3, 5, 7\} \subseteq Z_{12}$, B is not a semigroup and B does not contain any proper subsemigroup. Now P is a strong set ideal only over the group $G_5 = \{1, 11\}$ and not over any other group of Z_{12}.

We define these strong set ideals as special or unique strong set ideal over the group G. However T and P are not special strong set ideals of Z_{12}.

In view of this we have the following theorem.

THEOREM 3.4: *Let P be a special strong set ideal of a semigroup S over the group $G \subseteq S$ ($P \subseteq S$). P is a strong set ideal of S. Every strong set ideal of S in general need not be a special strong set ideal of S.*

***Proof*:** One way of the proof follows from the definition. The other way is true from the above theorem.

Example 3.29: Let $Z_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the semigroup under product. $G = \{1, 9\} \subseteq Z_{10}$ is a group under product modulo 10.

$G_1 = \{2, 4, 6, 8\} \subseteq Z_{10}$ is a group with 6 as the identity given by the following table.

×	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

Take $G_2 = \{1, 3, 7, 9\} \subseteq Z_{10}$, G_2 is also a group with 1 as its multiplicative identity.

×	1	3	7	9
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

G_2 contains a subgroup $G = \{1, 9\} \subseteq Z_{10}$. G_1 contains $G_3 = \{6, 4\} \subseteq Z_{10}$ as a subgroup.

Take $W = \{2, 4, 6, 5, 8\} \in Z_{10}$, W is a strong set ideal over the group $G_2 = \{1, 3, 7, 9\} \subseteq Z_{10}$. Clearly $G_1 = \{2, 6, 4, 8\} \subseteq W$ and W is also a strong set ideal over the group G_1.

Consider $C = \{0, 5\} \subseteq Z_{10}$, C is a strong set ideal over both the groups $G_2 = \{1, 3, 7, 9\}$ and $G_1 = \{2, 6, 4, 8\}$ of Z_{10}.

Example 3.30: Let $S = Z_{13}$ be the semigroup under product. $G_1 = \{1, 12\} \subseteq Z_{13}$ is a group under product.

$G_2 = \{1, 3, 4, 9, 10, 12\} \subseteq Z_{13}$ is again a group of Z_{13}. $G_1 \subseteq G_2$. $M_1 = \{0, 2, 11\}$, $N_1 = \{2, 11\}$, $M_2 = \{0, 3, 10\}$, $N_2 = \{3, 10\}$, $M_3 = \{0, 4, 9\}$ $N_3 = \{4, 9\}$, $M_4 = \{0, 5, 8\}$, $N_4 = \{5, 8\}$, $M_5 = \{0, 6, 7\}$ and $N_5 = \{6, 7\}$ are strong set ideals over the group G_1. Infact $P_1 = \{4, 9, 0, 6, 7, 5, 8\} \subseteq Z_{13}$ is also a strong set ideal of Z_{13} over the group $G_1 = \{1, 12\}$.

Clearly $P_1 = \{0, 4, 9, 6, 7, 5, 8\} \subseteq Z_{13}$ is not a strong set ideal of Z_{13} over the group $G_2 = \{1, 3, 4, 9, 10, 12\} \subseteq Z_{13}$. $G_3 = \{1, 5, 8, 12\} \subseteq Z_{13}$ be the group given by the following table.

×	1	5	8	12
1	1	5	8	12
5	5	12	1	8
8	8	1	12	5
12	12	8	5	1

Both the groups G_2 and G_3 contain $G_1 = \{1, 12\} \subseteq Z_{13}$. However P_1 is not a strong set ideal of Z_{13} over the group $G_3 = \{1, 5, 8, 12\}$.

THEOREM 3.5: *Let Z_p (p a prime) be the semigroup, Z_p has subgroups.*

The proof is direct and exploits simple number theoretic techniques.

Interested reader can study the strong set ideals and special strong set ideals of a semigroup S over a group $G \subseteq S$.

Now we proceed onto describe four types of ideals.

Let S be a semigroup P be a subsemigroup of S. Suppose M is another subsemigroup of S different from P and if for every $p \in P$ and $m \in M$, $mp, pm \in P$ we call P a one way subsemigroup ideal of S over the subsemigroup M of S.

If we have for every $p \in P$ and $m \in M$, mp and $pm \in M$ and P a one way semigroup ideal of S and M is also a one way semigroup ideal of S then we define (P, M) or (M, P) to be the two way subsemigroup ideal of S.

We will first illustrate this situation by some examples.

Example 3.31: Let $Z_6 = S$ be the ring of integers.

$P = \{0, 3, 5, 1\}$ and $M = \{0, 2, 4\}$ are subsemigroups of S.

Clearly M is a one way subsemigroup ideal group over P. However P is a not a one way subsemigroup ideal over M. Suppose we take $P_1 = \{0, 3\}$ then (P_1, M) $((M,P_1))$ is a two way semigroup ideal of S.

Example 3.32: Let S = {0, 1, …, 15} = Z_{16} the semigroup under product. {0, 1, 15} = P is a subsemigroup. M = {0, 2, 4, …, 14} $\subseteq Z_{16}$ is again a subsemigroup. M is a one way subsemigroup ideal over the subsemigroup P of S; we see P is not a one way subsemigrouip ideal over the subsemigroup M of S.

Example 3.33: Let S = {0, 1, 2, …, 25} = Z_{24} be the semigroup under product. P = {0, 6, 12} $\subseteq$ S is a subsemigroup of S. M = {4, 0, 8, 16} $\subseteq$ S is again a subsemigroup of S. (P, M) is a two way subsemigroup ideal of S. If T = {0, 1, 23} $\subseteq$ Z. T is not a subsemigroup ideal over P or M.

THEOREM 3.6: *Let S be a semigroup with unit 1. Suppose P is a subsemigroup with 1 and M is another semigroup different from P. P is a one way subsemigroup ideal of S over M.*

***Proof*:** If P is a one way subsemigroup ideal of S over M and if $1 \in P$ clearly $1.m \in P$ for every $m \in M$ so $M \subseteq P$. Hence the claim.

Now we proceed onto define group-subsemigroup ideal of a semigroup over a subsemigroup. Suppose S be a semigroup, G a group of S, $G \subseteq S$. Let P be a subsemigroup of $G \cap P \neq P$ or G.

If for every $g \in G$ and $p \in P$, pg and gp $\in$ P. We call P to be a group-subsemigroup ideal over a subsemigroup $P \subseteq S$.

If G is such that gp and pg $\in$ G for all $g \in G$ and for all $p \in P$, we call G to be subsemigroup-group ideal of S over the group G.

We will illustrate both the situations by some examples.

Example 3.34: Let S = {0, 1, 2, …, 11} = Z_{12} be the semigroup under product. G_1 ={1, 11} ⊆ S is a group in S. G_2 = {4, 8} ⊆ S is a group in S. P_1 = {2, 4, 8, 0} ⊆ S is a semigroup under product. P_1 is a group-subsemigroup ideal of S over the groups G_1 as well as G_2. But $G_2 \subseteq P_1$ this is to be noted.

Consider P_2 = {0, 2, 4, 6, 8, 10} ⊆ S, P_2 is a subsemigroup of S; P_2 is a group-subsemigroup ideal over the group G_1.

Take P_3 = {0, 3, 9} ⊆ S, P_3 is a group-subsemigroup ideal of S over the groups G_1 and G_2. G_3 = {3, 9} ⊆ S is again a group in S. P_2 is again a group-subsemigroup ideal of S over the group G_3.

Example 3.35: Let S = {0, 1, 2, …, 15} be a group of S. P_1 = {0, 2, 4, 6, 8, 10, 12, 14} ⊆ S is a subsemigroup of S. Clearly P_1 is not a group.

G_2 = {1, 3, 9, 11} ⊆ S is a group; P_1 is a group-subsemigroup ideal over the groups G_1 and G_2 of S.

Consider P_2 = {0, 4, 8} ⊆ S is a subsemigroup of S. Clearly P_2 is a group-subsemigroup ideal over G_1 = {1, 15} and is not a group-semigroup ideal over G_2 = {1, 3, 9, 11} as 3.4 $\notin P_2$.

We on similar lines define the notion of subsemigroup-group ideal of a semigroup S.

Let G be a group such that G ⊆ S, P ⊆ S be a subsemigroup of S. P ∩ G ≠ G or P. We say G is a subsemigroup-group ideal of S if for every g ∈ G and p ∈ P, pg and gp ∈ G, P the subsemigroup of S.

We will illustrate this situation by some simple examples.

Example 3.36: Let S = {0, 1, 2, …, 6} be the semigroup. G = {2, 4} be group in S given by the following table.

×	2	4
2	4	2
4	2	4

and P = {0, 3, 4} $\subseteq$ S

be the subsemigroup given by the following table.

×	0	3	4
0	0	0	0
3	0	3	0
4	0	0	4

Clearly G is a subsemigroup-group ideal of the semigroup S over the subsemigroup P $\subseteq$ S.

Example 3.37: Let Z_5 = {0, 1, 2, 3, 4, 5} be a semigroup. G = {1, 2, 3, 4} is a group. If P = S then G is a semigroup-group ideal of the semigroup P = S over S.

This is trivial or we do not accept this structure as a group-subsemigroup ideal of S.

Example 3.38: Let S = {0, 1, 2, …, 11} be a semigroup under product modulo 12. Take G = {4, 8} $\subseteq$ S given by the following table.

×	4	8
4	4	8
8	8	4

is a group in S. P = {0, 3, 6, 9} $\subseteq$ S; be a subsemigroup under × given by the following table.

×	0	3	6	9
0	0	0	0	0
3	0	9	6	3
6	0	6	0	6
9	0	3	6	9

G is a subsemigroup-group ideal of S over P the subsemigroup.

Example 3.39: Let Z_{15} = {0, 1, 2, …, 14} be the semigroup under product. G = {5, 10} $\subseteq Z_{15}$ is a group of order two given by the following table.

×	5	10
5	10	5
10	5	10

with 10 as its identity.

P = {0, 3, 6, 9, 12} $\subseteq Z_{15}$ is a subsemigroup under product ×.

×	0	3	6	9	12
0	0	0	0	0	0
3	0	9	3	12	6
6	0	3	6	9	12
9	0	12	9	6	3
12	0	6	12	3	9

G is a subsemigroup-group ideal over the subsemigroup P of S.

Example 3.40: Let S = Z_{19} = {0, 1, 2, …, 18} be a semigroup under ×. G_1 = {1, 18} $\subseteq$ S is a group given by the following table.

×	1	18
1	1	18
18	18	1

Can S have subsemigroups?

Example 3.41: Let Z_7 be the semigroup under ×; G = {1, 2, 4} $\subseteq Z_7$ is a group under product. G_2 = {1, 6} $\subseteq Z_7$ is a group under product. Z_7 has no subsemigroups other than {0, 1, 2, 4} and {0, 1, 6}.

Example 3.42: Let Z_{10} be the semigroup under product G_1 = {1, 9} $\subseteq Z_{10}$ be the group. P = {0, 2, 4, 6, 5, 8} $\subseteq Z_{10}$ is a subsemigroup of Z_{10} is a subsemigroup-group of Z_{10} over the group G.

Example 3.43: Let Z_{14} = S = {0, 1, …, 13} be the semigroup. G = {1, 13} $\subseteq Z_{14}$ be the group.

P = {0, 2, 4, 6, 8, 10, 12, 7} $\subseteq Z_{14}$.

×	0	2	4	6	8	10	12	7
0	0	0	0	0	0	0	0	0
2	0	4	8	12	2	6	10	0
4	0	8	2	10	4	12	6	0
6	0	12	10	8	6	4	2	0
8	0	2	4	6	4	10	12	0
10	0	6	12	4	10	2	8	5
12	0	10	6	2	12	8	4	0
7	0	0	0	0	0	5	0	7

is not a subsemigroup of Z_{14}. However M = {2, 4, 6, 8, 10, 12} $\subseteq Z_{14}$ is a group of Z_{14} with 8 as its identity under ×.

T = {1, 7, 9, 11} $\subseteq Z_{14}$ is a subsemigroup of Z_{14} given by the following table.

×	1	7	9	11
1	1	7	9	11
7	7	7	7	7
9	9	7	11	11
11	11	7	1	9

T is not a subsemigroup-group ideal over the group G or M.

Now we proceed onto define the notion of group-group ideal of a semigroup S. Let S be a semigroup under ×. G be a group in S; G $\subseteq$ S and H be another group in S different from G. If GH = HG, then we say G is a group-group ideal of a semigroup S over the group H.

We give examples of them.

Example 3.44: Let S (3) = {The set of all mappings of the set (1, 2, 3) to (1, 2, 3)}. S (3) is the symmetric semigroup.

Consider $G = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}\right\} \subseteq S(3)$ and

$H = \left\{\begin{pmatrix}1 & 2 & 3\\1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}\right\} \subseteq S(3)$ be groups in S(3).

GH = HG, that is G a group-group ideal of the semigroup S over the group H.

Example 3.45: Let S(4) = {the set of all mappings of the set (1, 2, 3, 4) to (1, 2, 3, 4)} be the symmetric semigroup. $A_4 \subseteq S(4)$ is a group in S(4).

Consider $G = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \right.$

$\left. \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix} \right\} \subseteq S(4)$ is again a semigroup. We see $A_4G = GA_4$ so A_4 is a group-group ideal of the semigroup over the group G.

THEOREM 3.7: *Let S(n) be the symmetric semigroup. The group A_n in S(n) is such that $A_nG = GA_n$ for some group $G \in S(n) \setminus A_n$.*

The proof follows from the fact S(n) contains S_n the group and S_n has subgroups $H \subseteq S_n \setminus A_n$ such that $H A_n = A_nH$, $H \subseteq S(n) \setminus A_n$.

Hence the claim.

Now we can define S-set ideals and S-subsemigroup-group ideal and group-S-semigroup ideal of a semigroup S. All these are carried out as a matter of routine and interested reader is left with task of constructing all these types of ideals.

We proceed onto define the notion of minimum and maximum (minimal and maximal) set ideals of a semigroup and illustrate this situation by some examples.

Let S be a semigroup $S_1 \subseteq S$ be a subsemigroup of S. Suppose $P \subseteq S$ is a proper subset of S and P is a set ideal of S relative to the subsemigroup S_1 of S. We say P is a minimal set ideal of S relative to the subsemigroup S_1 of S if we have $\phi \neq P_1 \subseteq P \subseteq S$ and $|P_1| \geq 2$ and if P_1 is also a set ideal of S relative to the subsemigroup S_1 then either $P_1 = \phi$ or $|P| = 1$ or $P_1 = P$ then we call P the minimal set ideal of S relative to the subsemigroup S_1 of S.

Likewise we say the set ideal P of S relative to the subsemigroup S_1 of S is a maximal set ideal of S if $P \subseteq P_1 \subseteq S$

where P_1 is also a set ideal of S relative to S_1, then P = P_1 or P_1 = S; we call P to be the maximal set ideal of S relative to the subsemigroup S_1 of S. It is to be noted that we cannot define maximal or minimal with respect to some other subsemigroup S_1 of S.

Example 3.46: Let S = {0, 1, 2, 3, 4, 5} be the semigroup under product modulo 6. S_1 = {0, 2, 4} $\subseteq$ S is a subsemigroup of S. Take P = {0, 3} $\subseteq$ S. P is a set ideal of S over S_1.

Take B = {0, 5} $\subseteq$ S. B is also a set ideal of S over S_1.

We see both B and P are minimal set ideals of S over the subsemigroup S_1 of S. M = {0, 5, 3} $\subseteq$ S is a maximal set ideal of S over the subsemigroup S_1 of S. We have only one maximal set ideal for S over the subsemigroup S_1 of S.

Example 3.47: Let S = Z_{10} = {0, 1, 2, …, 9} be the semigroup of integers modulo 10. S_1 = {0, 1, 5, 6} $\subseteq$ S is a subsemigroup of S.

S_2 = {0, 2, 4, 6, 8, 1} $\subseteq$ S is also a subsemigroup of S. P_1 = {0, 2, 4, 6, 8} $\subseteq$ S is a maximal set ideal of S over the subsemigroup S_1 = {0, 1, 5, 6}.

P_2 = {0, 2} $\subseteq$ S is a minimal set ideal of S over the subsemigroup S_1.

P_3 = {0, 6} $\subseteq$ S is a minimal set ideal of S over the subsemigroup S_1.

P_4 = {0, 4} $\subseteq$ S is a minimal set ideal of S over the subsemigroup S_1.

However P_5 = {0, 2, 6} $\subseteq$ S is not a minimal or maximal set ideal of S over the subsemigroup S_1 of S.

THEOREM 3.8: *Let S be a semigroup. $S_1 \subseteq S$ be a subsemigroup of S. Let $P_1, P_2, \ldots, P_n$ be set ideals of S over the subsemigroup S_1.*

(1) $\bigcup_{i=1}^{n} P_i = P$ *is again set ideal of S over the subsemigroup* S_1.

(2) $\bigcap_{i=1}^{n} P_i = \phi$ *or {0} or P. P is a set ideal of S over the subsemigroup* S_1 *of S.*

It is important and interesting to note that in case of set ideal of S over a subsemigroup S_1 we get the union to be again a set ideal of S over S_1. However if the intersection is non empty or 0 or some P we see that is also a set ideal of S over S_1. However if we vary S_1 over which the set ideals are defined no meaningful result can be had.

In view of this we can have several nice properties both for rings as well as for semigroups.

THEOREM 3.9: *Let L denote the collection of all set ideals of a semigroup (or a ring) S (or R) over the subsemigroup* S_1 *(or* R_1 *the subring of R) of S. (L, ∪, ∩) is a lattice called the lattice of set ideals related to the subsemigroup* S_1 *(or* R_1 *the subring).*

***Proof*:** Let S be the semigroup $S_1 \subseteq S$ be a proper subsemigroup of S. Let $P_1, P_2, \ldots, P_n$ be set ideals of S over the subsemigroup S (or ideals of R over the subring R_1).

$P = \bigcup_{i=1}^{n} P_i$ and $T = \bigcap_{i=1}^{n} P_i$ $(= \phi$ or $\{0\}$, $T \neq \phi$ or $\{0\}\}$.

$L = \{\{P, T, P_1, \ldots, P_n\}, \cup, \cap\}$ is a lattice of set ideals relative to the subsemigroup S_1 of S (subring R_1 of R).

Infact $L = \left\{\bigcup_{i=1}^{n} P_i, P_1, P_2, \ldots, P_n, \bigcap_{i=1}^{n} P_i\right\}$ is a lattice with $\bigcap_{i=1}^{n} P_i$ as its least element as a set ideal over S_1 or ϕ and $\bigcup_{i=1}^{n} P_i$ is the maximal set ideal of S over the subsemigroup S_1 of S.

We will illustrate this situation by some examples.

Example 3.48: Let $S = Z_{14}$ be the semigroup under product $\times$. $S_1 = \{0, 1, 7\} \subseteq S$ is a subsemigroup of S. $P_1 = \{0, 2\} \subseteq S$ is a set ideal of S over the subsemigroup S_1 of S.

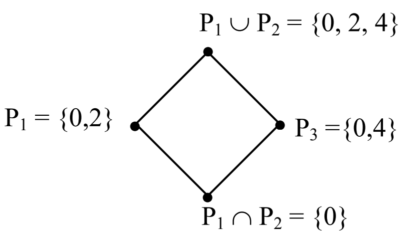
$P_2 = \{0, 4\} \subseteq S$ is a set ideal of S over the subsemigroup S_1 of S. $P_3 = \{0, 6\} \subseteq S$, $P_4 = \{0, 8\} \subseteq S$, $P_5 = \{0, 10\} \subseteq S$ and $P_6 = \{0, 12\} \subseteq S$ are set ideals of S over the subsemigroup S_1.

Consider $P_1 = \{0, 2, 4\} \subseteq S, \ldots, P_{21}\ \{0, 10, 12\} \subseteq S$ are all set ideals of S over S_1, $P_{22} = \{0, 2, 4, 6\} \subseteq S, \ldots, P_{41} = \{0, 8, 10, 12\} \subseteq S$ are set ideals of S over the subsemigroup S_1. $P_{42} = \{0, 2, 4, 6, 8\} \subset S$, $P_{43} = \{0, 2, 4, 6, 10\} \subseteq S, \ldots, P_{56} = \{0, 6, 8, 10, 12\} \subseteq S$ are all set ideals of S over S_1, $P_{57} = \{0, 6, 8, 10, 2, 4\} \subseteq S, \ldots, P_{62} = \{0, 4, 6, 8, 10, 12\} \subseteq S$ are set ideals of S over S_1. $P_{63} = \{0, 2, 4, 6, 8, 10, 12\} \subseteq S = \bigcup_{i=1}^{62} P_i$ is again a set ideal of S over S_1. Thus $L = \{(0), \bigcap_{i=1}^{62} P_i, P_1, P_2, \ldots, P_{62}, P_{63} = \bigcap_{i=1}^{62} P_i\}$ is a lattice of set ideals of S over the subsemigroup S_1.

Example 3.49: Let $S = Z_6 = \{0, 1, 2, 3, 4, 5\}$ be the semigroup under product. $S_1 = \{0, 3, 1\} \subseteq S$ be a subsemigroup of S given by the following table.

$\times$	0	1	3
0	0	0	0
1	0	1	3
3	0	3	3

Take $P_1 = \{0, 2\} \subseteq S$, $P_2 = \{0, 4\} \subseteq S$ and $P_3 = \{0, 2, 4\} \subseteq S$, $P_3 = P_1 \cup P_2$ are set ideals of S over the subsemigroup S_1, $L = \{(0), P_1 \cap P_2, P_1, P_2, P_1 \cup P_2 = P_3\}$ is a lattice with the following diagram.



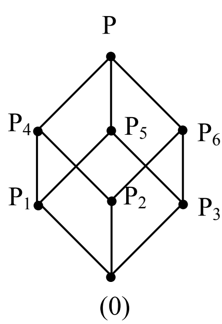
L is a distributive lattice infact a Boolean algebra of order four.

Example 3.50: Let $Z_8 = S = \{0, 1, 2, \ldots, 7\}$ be the semigroup. $S_1 = \{0, 1, 7\}$ be the subsemigroup of S.

Let $P = \{0, 2, 6, 3, 5, 4\}$ be a set ideal over the subsemigroup S_1 of S.

$P_1 = \{0, 2, 6\} \subseteq S$, $P_2 = \{0, 4\} \subseteq S$, $P_3 = \{0, 3, 5\} \subseteq S$, $P_4 = \{0, 2, 6, 4\} \subseteq S$, $P_5 = \{0, 2, 6, 3, 5\} \subseteq S$ and $P_6 = \{0, 3, 5, 4\} \subseteq S$ be set ideals of S over the subsemigroup S_1.

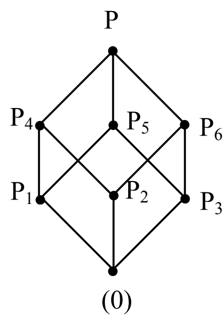
Let $L = \{\{0\}, P_1, P_2, P_3, P_4, P_5, \bigcap_{i=1}^{6} P_i = P\}$ be lattice of set ideals.



L is a distributive lattice infact a Boolean algebra of order 8.

Example 3.51: Let S = {Z_9} = {0, 1, 2, 3, 4, 5, 6, 7, 8} be the semigroup under product. P = {0, 2, 7, 3, 6, 4, 5} be the set ideal of S over the subsemigroup S_1 = {0, 1, 8} $\subseteq$ S = Z_9.

P_1 = {0, 2, 7}, P_2 = {0, 3, 6}, P_3 = {0, 4, 5}, P_4 = {0, 2, 7, 3, 6}, P_5 = {0, 2, 7, 4, 5}, P_6 = {0, 3, 6, 4, 5} be set ideals of S over the subsemigroup S_1 of S. L = {(0) = $\bigcap_{i=1}^{6} P_i$, P_1, P_2, …, P_6, P = $\bigcup_{i=1}^{6} P_i$ } be the lattice of set ideals of S over S_1.



L is a distributive lattice of order 8.

Example 3.52: Let S = Z_9 = {0, 1, 2, …, 8} be the semigroup under product. S_1 = {1, 2, 4, 5, 7, 8} $\subseteq$ S, be the subsemigroup of S. P = {0, 3, 6} be the set ideal of S over S_1. Clearly we see this is the only set ideal of S over S_1.

Example 3.53: Let S = (Z, $\times$) be the semigroup. Let {0, 1, –1} = $S_1 \subseteq$ S be the subsemigroup of S. 2Z = P_1, 3Z = P_2, 4Z = P_3, …, nZ = P_{n-1} are all set ideals of S over the subsemigroup S_1 of S.

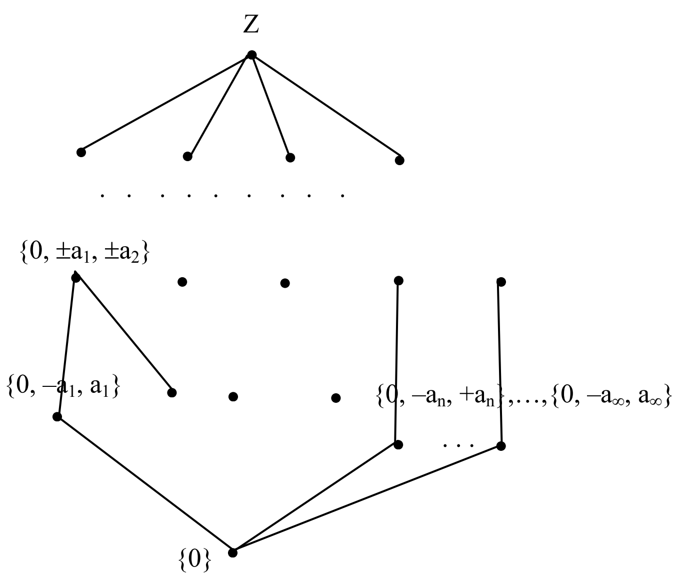
Not only these N_1 = {–2, 2, 0, 198, –198} $\subseteq$ S is also a set ideal of S over S_1. Thus the collection of set ideals of S over the subsemigroup S_1 is infinite in number; {0} is the least element

and Z is the greatest element of this infinite lattice. For instance $M_1 = \{1, 2, 3, 0, -1, -2, -3\}$ is also a set ideal of S over S_1.

$M_2 = \{2, -2, 0\}$ is also a set ideal of S over S_1 and so on. However if P is a set ideal of S over the subsemigroup S_1 then minimum P is, $|P| = 3$ and $P = \{0, -n, n\}$; n an integer in Z.

Any other P_2 of order five alone can occur that is with elements of the form $P_2 = \{0, a, -a, b, -b\}$, $a, b \in Z$ is a set ideal of Z over the subsemigroup $S_1 = \{0, 1, -1\}$. Next P_3 will be of order seven and so on. Any P_t would be of odd order say 2n+1 with $\pm a_1, \pm a_2, \ldots, \pm a_n \in Z$.

Thus we have a collection and we have a layer say 2Z, 3Z, 5Z, …, pZ, p's prime then a layer with $\langle p, q \rangle \subseteq Z$, p and q two distinct primes and reach Z. Z being the greatest element and {0} the smallest element.



Thus $\{0, a_i, -a_i\}$ are the atoms of this lattice. Now L is a lattice of infinite order.

Example 3.54: Let S = (Q, ×) be the semigroup under ×. $S_1 = \{0, 1, -1\} \subseteq S$ be the subsemigroup of S.

Let M be the set ideals of S associated with the subsemigroup S_1. Like in case of example 3.53 all set ideals are of odd order and order three set ideals form the atoms of M.

$\{M, \cap \cup\}$ is a lattice of infinite order.

L in example 3.53 is a sublattice of M.

Example 3.55: Let S = (R, ×) be the semigroup under product $S_1 = \{0, -1, 1\}$ be a subsemigroup of S. Let N denote the lattice of all set ideals of S over the subsemigroup S_1 of S. N is also an infinite lattice and atoms of N are set ideals of S over S_1 of order three. Clearly L in example 3.53 and M in example 3.54 are sublattices of N. $L \subseteq M \subseteq N$.

Finally we use, C the complex field, to build set ideals using c. Here we have two subsemigroups of C of finite order. $S_1 = \{0, 1, -1\}$ and $S_2 = \{0, -1, 1, i, -i\}$ in C.

Using these two finite subsemigroups we can build set ideals over S_1 or S_2.

All set ideals over S_1 need not be contained in the collection of set ideals over S_2 however the reverse is true.

Thus we can get two lattices G_1 and G_2 associated with the subsemigroups S_1 and S_2 respectively where G_2 will be a sublattice of G_1. Thus we have seen the lattice of set ideals of a semigroup S defined over a subsemigroup S_1 of S both finite and infinite.

Thus the set ideals of a semigroup S built over the same subsemigroup S_1 of S is an infinite lattice.

Now we give a topological structure to these set ideals.

DEFINITION 3.7: *Let S be a semigroup. Let T denote the collection of all set ideals of S relative to a subsemigroup S_1 of S. (T, S_1) is defined as a topological set ideal space of S relative to the subsemigroup S_1 of S.*

The following facts are to be observed:
(i) T contains either {0} or the empty set.
(ii) T itself is a set ideal of S relative to the subsemigroup S_1 of S.
(iii) The union of any family of set ideals of S relative to the subsemigroup S_1 of S is again a set ideal of S over S_1.
(iv) Intersection of any two hence even infinite number of set ideals of S over the subsemigroup S_1 of S is again a set ideal of S over S_1 or ϕ or {0}.

Thus T with the property of set ideals of a semigroup S over the subsemigroups S_1 of S is a topological space T relative to the subsemigroup S_1 of the semigroup S.

We will illustrate this situation by some example.

Example 3.56: Let S = (Z, ×) be a semigroup under product. $S_1 = \{0, 1, -1\} \subseteq S$ be a subsemigroup of S. T be the collection of set ideals of S over the subsemigroup S_1 of S. (T, S_1) is a topological space of set ideals of S over the subsemigroup S_1.

Clearly $P = \{(0, a_i, -a_i) \mid a_i \in Z, 1 \leq i \leq \infty\} \subseteq T$ is a basic set ideals of the space T over the subsemigroup S_1. Clearly T satisfies both the first and second axiom of countability. Further as T satisfies the second axiom of countability, T is separable.

Example 3.57: Let S = {Q, ×} be a semigroup under product. Let $S_1 = \{0, 1, -1\}$ be the subsemigroup of S. Let P = {all set ideals of S over the subsemigroup S_1 of S}. P is a topological space of set ideals of a semigroup S over the subsemigroup S_1 of S. Infact $T \subseteq P$ is a subspace of set ideals of S over the subsemigroup S_1 of S (T mentioned in example 3.56). Further all set ideal topological space properties enjoyed by T is

enjoyed by the set ideal topological space P over the subsemigroup S_1 of S.

Example 3.58: Let S = (R, ×) be the semigroup. $S_1 = \{0, -1, 1\} \subseteq (R, \times) = S$ be a subsemigroup of S. Find the set ideal topological space P of S over the subsemigroup S_1. P = {all set ideals of S defined over the subsemigroup S_1 of S}.

Example 3.59: Let S = (C, ×) be the semigroup of complex numbers. Suppose $S_1 = \{-1, 1, 0\} \subseteq S$ be a subsemigroup of S. Study the topological space V of set ideals over the subsemigroup $S_1 \subseteq S$. If $S_2 = \{-1, 1, 0, -i, i\} \subseteq S$ be a subsemigroup of S and if W = {collection of all set ideals of S over the subsemigroup S_2 of S}, W be the topological space of set ideals of S over the subsemigroup S_2 of S.

Example 3.60: Let $\langle R \cup I \rangle = S$ be the semigroup $S_1 = \{1, I, 0\}$ be a subsemigroup of S under ×. Let P = {collection of set ideals of S over the subsemigroup S_1 of S} be the neutrosophic set ideal topological space of S over the subsemigroup S_1 of S.

Suppose $S_2 = \{1, -1, 0, I, -I\}$ be the subsemigroup of S. Let M = {collection of a set ideals of S over the subsemigroup S_2 of S} be the neutrosophic set ideal topological space of S over the subsemigroup S_2 of S study if $P \subseteq M$ or $M \subseteq P$?

Example 3.61: Let $S = \{\langle Z \cup I \rangle, \times\}$ be a semigroup. $S_1 = \{0, 1, I\} \subseteq S$ be a subsemigroup of S. Let P_1 = {collection of all set ideals of S over the subsemigroup S_1 of S} be a neutrosophic set ideal topological space of S the semigroup S over the subsemigroup S_1 of S.

Take $S_2 = \{0, 1, I, -I, -1\} \subseteq S$ to be a subsemigroup of S. Let P_2 = {collection of all set ideals of S over the subsemigroup S_2 of S} be the neutrosophic set ideal topological space of S over the subsemigroup S_2 of S.

Example 3.62: Let $S = \{\langle C \cup I\rangle\}$ be semigroup under product. $S_1 = \{0, I, -I\} \subseteq S$ be a subsemigroup of S. Let M = {collection of all set ideals of S over the subsemigroup S_1 of S} be the neutrosophic complex set ideal topological space of S over the subsemigroup S_1 of S. $M = \{\{0, a_1I, -a_1I\} \cup \{0, a_1I, -a_1I, a_2I, -a_2I \mid \pm a_1, \pm a_2 \in C\} \cup \ldots \cup \{0, \pm a_1I, \pm a_2I, \pm a_3I, \ldots, \pm a_nI, \ldots \mid$ where $a_j \in C; 1 \leq j \leq n\}$. Find the lattice associated with the set ideals of S relative to the subsemigroup S_1 of S.

Example 3.63: Let $S = \{Q \cup I\}$ be a neutrosophic semigroup under $\times$.

Consider $S_1 = \{I, -I, 0, 1, -1\} \subseteq S$ be the subsemigroup of S. P = {collection of all set ideals of S over the subsemigroup S_1 of S} is a neutrosophic set ideal topological space of S relative to S_1 of S. Also P enjoys a neutrosophic lattice structure with 0 as its least element and $P_i = \{0, a_iI, -a_iI\} \mid \pm a_i \in Q$ as atoms for atoms of P can have the minimum cardinality to be three. For instance $\{0, 7+I, -7 - I\}$ is not a set ideal of S over the subsemigroup S_1 of S.

Thus we can have complex set ideal topological space over a subsemigroup $S_1 \subseteq (C, \times)$, we have also associated with the complex set ideal topological space a complex set ideal lattice of infinite order.

Likewise we have $\langle Z \cup I\rangle$ to contribute to integer neutrosophic topological space of set ideals over subsemigroups $S_1 = \{0, 1, -1\}$ or $S_2 = \{0, I, -I\}$ or $S_3 = \{0, \pm 1, \pm I\}$ of the semigroup $S = \{\langle Z \cup I\rangle, \times\}$. This integer neutrosophic topological space of set ideals can also be given the lattice structure of infinite order relative to every one of these subsemigroups S_1, i = 1, 2, 3.

Using $S = \{\langle Q \cup I\rangle, \times\}$ as a semigroup we can define rational number integer topological space of set ideals of S relative to the subsemigroup S_1, S_2 or S_3 of S mentioned earlier. These neutrosophic topological set ideals have a lattice

associated with it. Infact this lattice as well as the neutrosophic topological space associated with it are bigger than the ones associated with $\{\langle Z \cup I\rangle, \times\}$. Suppose instead of $\langle Q \cup I\rangle$ in S use $(\langle R \cup I\rangle, \times)$ we get for the same three subsemigroups S_1, S_2, S_3 three neutrosophic set ideal topological spaces of $S = (\langle R \cup I\rangle, \times)$ over the subsemigroups. These topological spaces contain the topological spaces built using $\langle Z \cup I\rangle$ and $\langle Q \cup I\rangle$. Finally if $\langle R \cup I\rangle$ is replaced by $\langle C \cup I\rangle$ in S that is $S = \{\langle C \cup I\rangle, \times\}$ is the semigroup. S_1, S_2 and S_3 are subsemigroups of S. We get three sets of set ideal topological spaces of $S = \{\langle C \cup I\rangle, \times\}$ relative to the three subsemigroup S_1, S_2 and S_3. These set ideal topological spaces contain the set ideal topological space related to the semigroups $(\langle Z \cup I\rangle, \times)$, $(\langle Q \cup I\rangle, \times)$ and $(\langle R \cup I\rangle, \times)$.

We can also have finite neutrosophic modulo integer set ideal topological spaces.

Example 3.64: Let $S = \{\langle Z_{12} \cup I\rangle, \times\}$ be a finite semigroup. $S_1 = \{0, 1, I\} \subseteq S$ be a subsemigroup of S. Suppose T = {all neutrosophic set ideals of S over the subsemigroup $S_1 \subseteq S$}; be the neutrosophic set ideal topological space of S over the subsemigroup S_1 of S.

T also has a lattice associated with it.

Example 3.65: Let $S = \{\langle Z_4 \cup I\rangle, \times\}$ be the neutrosophic semigroup. Take $S_1 = \{0, 1, 3\} \subseteq S$ to be a subsemigroup of S.

Consider T = {collection of all neutrosophic set ideals of S over the subsemigroup S_1 of S}. $\{0\} \in T$ is the least element of T. $\{0, 2\} \in T$, $\{0, 2I\} \subseteq T$ and $\{0, 2, 2I\} \in T$.

S = {0, 1, 2, 3, I, 2I, 3I, 1+I, 2+I, 3+I, 1+2I, 2+2I, 2+3I, 3+I, 3+2I, 3+3I} be the semigroup under $\times$.

T = {{0}, {0, 2}, {0, 2I}, {0, 2, 2I}, {0, I, 3I}, {0, 2, I, 3I}, {0, 2, 2I, I, 3I}, {2+2I, 0} and so on}.

Thus T the neutrosophic set ideal topological space of the semigroup S over the subsemigroup $S_1 = \{0, 1, 3\} \subseteq S$.

The lattice of neutrosophic set ideal topological space of T is of finite order.

Thus we can get neutrosophic finite set ideal topological space over any one of the subsemigroups S_i of S.

Example 3.66: Let $S = \{\langle Z_5 \cup I\rangle\}$ be a neutrosophic semigroup. $S_1 = \{4I, I, 0\} \subseteq S$ be a subsemigroup of S. Let T = {collection of all set ideals of S over the subsemigroup S_1 of S}. T is a neutrosophic set ideal topological space of S over the subsemigroup S_1 of S.

Example 3.67: Let $S = \{\langle C(Z_3) \cup I\rangle\}$ be the neutrosophic complex modulo integer semigroup. Let $S_1 = \{0, 1, 2\} \subseteq S$ be a subsemigroup. T = {collection of all set ideals of S over the subsemigroup S_1 of S}; T is a finite neutrosophic set ideal topological space of S over the subsemigroup S_1 of S.

Example 3.68: Let $M = \{\langle C(Z_{10}) \cup I\rangle\}$ be the neutrosophic complex modulo integer semigroup. Take $S_1 = \{0, 5, 5I\} \subseteq M$ to be a subsemigroup of M.

T = {set of all set ideals of M relative to the subsemigroup S_1 of M} is a neutrosophic complex modulo integer topological set ideal space of M over the subsemigroup S_1 of M of finite order.

T is also a lattice with {0} as the minimal element and $\bigcup_i B_i$; $B_i \in T$ is a maximal element. Now we can instead of working with finite complex numbers $C(Z_n)$ or finite neutrosophic complex numbers $\langle C(Z_n) \cup I\rangle$ as semigroups we can work with them as rings using appropriate subrings.

This will be described and illustrated in the following. Let Z or Q or R or C be a ring. Consider a subring in them say

$S_1 = \{nZ\} = \{0, \pm n, \pm 2n, \ldots, \infty\} \subseteq Z$ (or Q or R or C) is a subring. Now find T the collection of set ideals of the ring Z (or Q or R or C) related to this subring S_1 of Z we are not in a position to define topology on T. Though {0} is a set ideal in T.

$X = \bigcup_i X_i$; $X_i \in T$ is a set ideal in T relative to S_1; that is the union of any family T itself is a set ideal relative to S_1. Intersection of any pair is a set ideal is a set ideal relative to S_1. Thus T cannot be topologised with a set ideal topology relative to the subring S_1 of Z (or R or Q or C) as we do not have a basic set which can generate T; we call T to be a pseudo set ideal topological space of the ring Z (or R or Q or C) relative (over the subring) S_1 of Z (or R or Q or C).

We will illustrate this situation by some examples.

Of course we demand for every X_i in T; $S_i \subseteq X_i$.

Example 3.69: Let Z be the ring of integers. $S_1 = \{2Z\} = \{0, \pm 2, \pm 4, \pm 6, \ldots, \infty\}$ be the subring of Z. T = {all set ideal of Z relative to the subring S_1 of Z}.

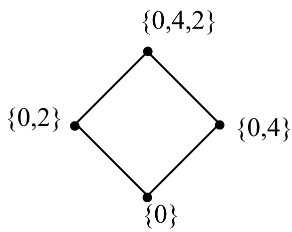
Here the minimal elements are assumed as m_iZ; m_i a very large integer, as Z has no minimal ideals and we have a pseudo topology defined on T over the subring S_1 of Z. T is a set ideal pseudo topological space of Z over the subring S_1 of Z. Infact T cannot be given the lattice structure where the minimal elements cannot be pronounced as Z has no minimal ideals; of course the least or minimal element of T is {0} and Z is the greatest element, so only we name it as a pseudo topological space.

Now we can define set ideal topological space of the ring Z_n ($n < \infty$) over a subring S_1 of Z_n in a similar way. These are not pseudo set ideal topological spaces.

We give some examples of them.

Example 3.70: Let R = {Z_6, ×, +} be the ring. S_1 = {0, 3} ⊆ R is a subring of S. The set ideals T of R relative to the subring S_1 are as follows T = {{0}, {0, 2}, {0, 4}, {0, 2, 4}}. Now T is a set ideal topological space of the ring Z_6 over the subring S_1 of Z_6.

The lattice representation of T is as follows:



Now if we in our definition of set ideals include 3 also as a subset of Z_6 which can contribute to set ideals then we have {0, 3} ∈ T_1 (T_1 is the new collection of set ideals of Z_6 over the subring S_1; that this collection can contain {0, 3} as a subset also).

Thus T_1 = {{0}, {0, 2}, {0, 4}, {0, 3}, {0, 2, 3}, {0, 4, 3}, {0, 2, 4} {0, 4, 2, 3}, {0, 1, 2, 3}, {0, 1, 4, 3}, {0, 1, 2, 3, 4}, {0, 5, 3}, {0, 5, 2, 3}, {0, 5, 4, 3}, {0, 5, 4, 2, 3}, {0, 1, 5, 2, 3}, {0, 1, 5, 3, 4}, {0, 1, 5, 2, 4, 3} = Z_6}.

Thus T_1 is a set ideal topological space of Z_6 relative to the subring {0, 3}.

Clearly T ⊆ T_1.

We can have lattice representation for both and the lattice related with T is contained in the lattice related with T_1. We can use S_2 = {0, 1, 5} ⊆ Z_6 as a subsemigroup and get yet two other topological spaces different from T_1 and T_2. Let P_1 = {collection of all set ideals of Z_6 relative to the subsemigroup S_2} = {{0}, {0, 2}, {0, 4}, {0, 4, 2}, {0, 5}, {0, 2, 5}, {0, 4, 5}, {0, 4, 2, 5}, {0, 3}, {0, 3, 5}, {0, 3, 2}, {0, 3, 4}, {0, 3, 4, 2}, {0, 3, 2, 5}, {0, 3, 4, 5}, {0, 3, 4, 2, 5}}.

We see P_1 is a set ideal topological space of Z_6 relative to the subsemigroup S_2 of Z_6. We can also get the lattice related with the set ideal topological space of the semigroup Z_6 relative to the subsemigroup S_2 of Z_6.

Next we proceed onto define several types of set ideal topological spaces like Smarandache set ideal topological spaces of the ring R, over a subring S of R, Smarandache quasi set ideal topological space of a ring over a subring of R.

Smarandache strongly quasi set ideal topological space of the ring R over the subrings S_1 and S_2 of R, Smarandache perfect set ideal topological space of a ring R, Smarandache simple perfect set ideal topological ring, Smarandache prime set ideal topological space of the ring R and prime set ideal topological space over the subring S of R.

We also have all these structures in case of semigroups.

We will define and illustrate them with examples.

DEFINITION 3.8: *Let S be a semigroup. S_1 be a subsemigroup of S. $P \subseteq S$ be a prime set ideal of S over S_1. Let T = {collection of all prime set ideals of S over S_1}; T is a topological space of set ideals over the subsemigroup S_1 of the semigroup S. We define T to be a prime set ideal topological space of S over the subsemigroup S_1 of S.*

Example 3.71: Let $S = Z_{12}$ be the semigroup under $\times$. Let $\{0, 1, 11\} = S_1$ be a subsemigroup of S. Take $P_1 = \{0, 3, 9\} \subseteq S$; P_1 is a prime set ideal of S over S_1 we call singletons other than (0) as trivially prime set ideals. $P_2 = \{0, 2, 10\}$ is a prime set ideal of S over S_1. $P_3 = \{0, 3, 9, 6\}$ is again a prime set ideal of S over S_1. $P_4 = \{0, 2, 10, 6\} \subseteq S$ is also a prime set ideal of S over S_1. Now $P_5 = \{0, 3, 9, 2, 10\} \subseteq S$ is also a prime set ideal of S over S_1. Finally $P_6 = \{0, 3, 9, 2, 10, 6\} \subseteq S$ is also a prime set ideal of S over S_1 and so on.

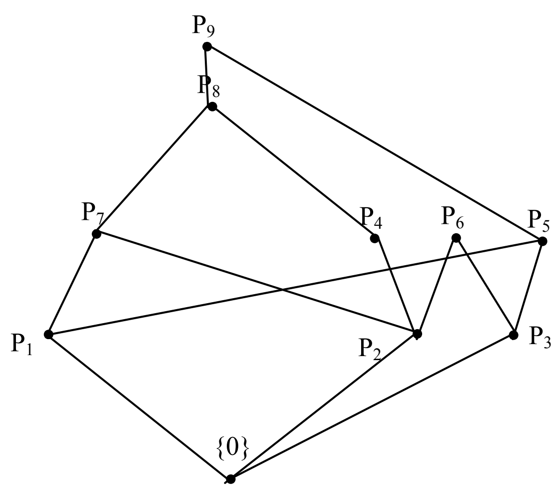
Thus $\{\{0\}, P_1, P_2, \ldots, P_6, \ldots\}$ form a prime set ideal topological space of S over the subsemigroup S_1 of S of finite order.

Example 3.72: Let $S = Z_{10}$ be the semigroup under product. $S_1 = \{0, 1, 9\}$ be a subsemigroup of S.

$P_0 = \{0\}$, $P_1 = \{0, 5\}$, $P_2 = \{0, 2, 8\}$, $P_3 = \{0, 3, 7\}$, $P_4 = \{0, 2, 8, 4, 6\}$, $P_5 = \{0, 3, 7, 5\}$, $P_6 = \{0, 2, 8, 3, 7\}$, $P_7 = \{0, 2, 8, 5\}$, $P_8 = \{0, 2, 8, 4, 6, 5\}$ and $P_9 = \{0, 2, 3, 4, 5, 6, 7, 8\}$ be subsets of S.

Thus $T = \{P_0, P_1, P_2, \ldots, P_9\}$ is a prime set ideal topological space of S over the subsemigroup S_1 of S.

The lattice $(T, \cup, \cap)$ is as follows:



Clearly T is not a Boolean algebra. However $\{P_1, P_2, P_3, P_4\}$ acts as the basic set of the topological space T.

Example 3.73: Let $S = Z_8 = \{0, 1, 2, \ldots, 7\}$ be the semigroup under product. $S_1 = \{0, 1, 7\}$ be the subsemigroup of S. $T = \{P_0 = \{0\}, P_1 = \{0, 3, 5\}, P_2 = \{0, 2, 6\}, P_3 = \{0, 2, 6, 4\}, P_4 = \{0, 2, 6, 3, 5\}, P_5 = \{0, 3, 5, 2, 6, 4\}\}$ be a topological prime set ideal space of the semigroup over the subsemigroup S_1 of S.

Example 3.74: Let S = Z_9 = {0, 1, 2, …, 8} be the semigroup. S_1 = {0, 1, 8} be the subsemigroup of S. T = {P_0 = {0}, P_1 = {0, 2, 4, 6}, P_2 = {0, 2, 4}, P_3 = {0, 2, 6}, P_4 = {2, 7, 0}, P_5 = {0, 2, 6, 7}, P_6 = {0, 2, 4, 7}, …} be the prime set ideal topological space of S over S_1 of finite order.

These are different from the usual set ideal topological spaces of semigroups over the subsemigroup S_1 of S.

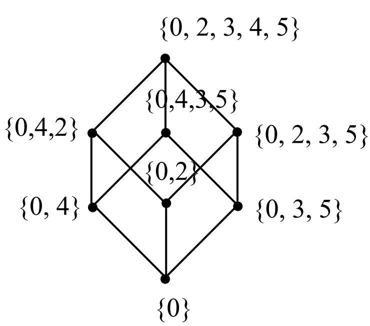
THEOREM 3.10: *Let S be a semigroup. S_1 a subsemigroup of S. T = {collection of all set prime ideals of S over the subsemigroup S_1}; be the set prime ideal topological space of S over the subsemigroup S_1 of S. T is a set ideal topological space of S over the subsemigroup S_1 of S. Conversely if T is a set ideal topological space of S over the subsemigroup S_1 of S then T in general is not a set prime ideal topological space of S over the subsemigroup S_1 of S.*

Proof: One way is true by the very definition. To prove the other claim we can only give examples.

Consider S = Z_6 the semigroup under product.

S_1 = {0, 3, 1} be the subsemigroup of S. Take T = {(0), {0, 4}, {0, 2}, {0, 4, 2}, {0, 3, 5}, {0, 4, 3, 5}, {0, 2, 3, 5}, {0, 4, 2, 3, 5}}, the topological space of set ideals of S over the subsemigroup S_1 of S.

The lattice associated with T is as follows:

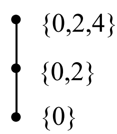


Clearly T is not a prime set ideal topological space of S over S_1. Hence the theorem.

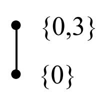
Next if we take a ring we can also have a prime set ideal topological space of the ring over a subring. The definition of this concept is similar to that of semigroups. So we only illustrate this situation by an example.

Example 3.75: Let S = Z_6 be a ring. S_1 = {0, 3} is a subring of Z_6. Let T = {{0}, {0, 2, 4}, {0, 2}} be the set prime ideal topological space of Z_6 over the subring S_1 of Z_6.

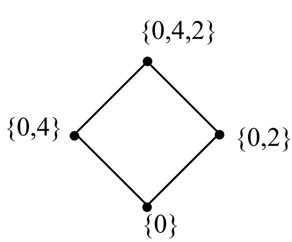
The lattice associated with T is as follows:



Now take M = {0, 2, 4} $\subseteq$ Z_6 the subring of Z_6. T_1 = {{0}, {0, 3}} is the prime set ideal topological space of Z_6 over the subring M of S associated with T_1. The lattice of T_1 is as follows:



Now take P_1 = {{0}, {0, 2}, {0, 4}, {0, 4, 2}} to be the set ideal topological space of Z_6 over the subring S_1 = {0, 3}. The lattice associated with P_1 is as follows:



The lattice is a Boolean algebra of order four where as the lattice of prime set ideal topological space over the same subring {0, 3} is only a distributive lattice or a chain lattice of order three. Thus it is to be noted the lattice associated with the prime set ideal topological space in general is not a Boolean algebra.

Now we proceed onto define Smarandache set ideal topological space of a ring (or a semigroup) over a subring (or a subsemigroup).

DEFINITION 3.9: *Let S be a semigroup. S_1 be a subsemigroup of S. T = {all set ideals P of S over the subsemigroup S_1 such that $S_1 \subseteq P$}; T is a defined as the Smaradache set ideal topological space of the semigroup S over (relative) to the subsemigroup S_1 of S.*

We can replace the semigroups by rings and in that case we call the set ideal topological space as Smarandache set ideal topological space of the ring over the subring of the ring.

We will illustrate both the situations by some examples.

Example 3.76: Let S = {0, 1, 2, …, 11} = Z_{12} be the semigroup under product. Let S_1 = {0, 6} $\subseteq$ S be a subsemigroup of S. T = {{0, 6}, {0, 2, 6}, {0, 4, 6}, {0, 3, 6}, {0, 5, 6}, {0, 7, 6}, {0, 8, 6}, {0, 9, 6}, {0, 10, 6}, {0, 6, 11}, {0, 1, 6}, {0, 6, 2, 4}, …, Z_{12}} be the topological space of Smarandache set ideals of Z_{12} over the subsemigroup S_1 = {0, 6}.

Now consider T_1 = {{0}, {0, 2}, {0, 4}, {0, 8}, {0, 10}, {0, 2, 4}, …, {0, 2, 4, 8, 10}}. T_1 is a set ideal topological space of Z_{12} over the subsemigroup S_1 = {0, 6} of Z_{12}.

We have the following interesting theorem; the proof of which is left as an exercise to the reader.

THEOREM 3.11: *Let S be a semigroup. S_1 a subsemigroup of S. T = {collection of all Smarandache set ideals of S over the subsemigroup S_1 of S}, be the Smarandache set ideal topological space of S over S_1. If T_1 is the set ideal topological space of S over the same subsemigroup S_1 of S. Then T_1 in general is not a S-set ideal topological space of S over S_1. Further T is always a set ideal topological space of S over S_1.*

Next we prove we have a class of Smarandache set ideal topological space of S over a subsemigroup $S_1 \subseteq S$.

THEOREM 3.12: *Let $S = Z_{2p}$ (p a prime or otherwise) be a subsemigroup of S. $S_1 = \{0, p\} \subseteq S$ is a subsemigroup of S. $T = \{\{0, p\}, \{0, p, 1\}, \{0, p, 2\}, \ldots, \{0, p, 2p-1\} \ldots, Z_{2p}\}$ is a Smarandache set ideal topological space of S over the subsemigroup S_1 of S.*

The proof is straight forward, hence left as an exercise to the reader.

Example 3.77: Let $S = Z_{12}$ be the ring of modulo integers. $S_1 = \{0, 6\}$ be the subring of S. $T = \{\{0, 6\}, \{0, 2, 6\}, \ldots, \{0, 11, 6\}, \{0, 1, 6\}, \ldots, Z_{12}\}$ be the Smarandache set ideal topological space of the ring Z_{12} over the subring $\{0, 6\} = S_1$. It is to be noted the Smarandache set ideal topological space of Z_{12} as a ring or as a semigroup over the subring {0, 6} is the same as over the subsemigroup {0, 6} when S is considered as a semigroup.

Inview of this we have the following theorem, the proof of which is left as an exercise to the reader.

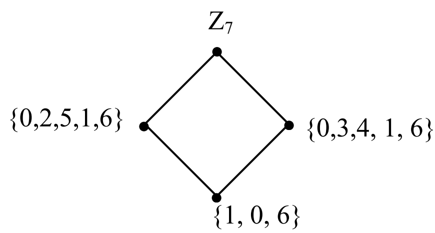
THEOREM 3.13: *Let $S = Z_{2p}$ be the ring of integers modulo 2p ($1 < p < \infty$). Then $S_1 = \{0, p\}$ be the subring of S. $T = \{\{0, p\}, \{0, 1, p\}, \{0, 2, p\}, \ldots, Z_{2p}\}$ is the Smarandache set ideal topological space of Z_{2p} over S_1 the subring.*

The above theorem has been proved for Z_{2p} as a semigroup. Infact we see Z_{2p} has the same topological S-set ideals as a

semigroup or as a ring. This is the interesting feature enjoyed by them.

Example 3.78: Let S = Z_7 be the semigroup. S_1 = {0, 1, 6} be the subsemigroup of S. T = {{0, 1, 6}, {0, 2, 5, 1, 6}, {0, 3, 4, 1, 6}, {0, 3, 4, 5, 2, 1, 6} = Z_7} is the Smarandache set ideal topological space of S over the subsemigroup S_1 = {0, 1, 6}.

The lattice of S-set ideals is as follows:



Clearly Z_7 is a ring which has no subrings.

THEOREM 3.14: *Z_p (p a prime) has no subrings hence no set ideals so no S-set ideals.*

The proof is left as an exercise to the reader.

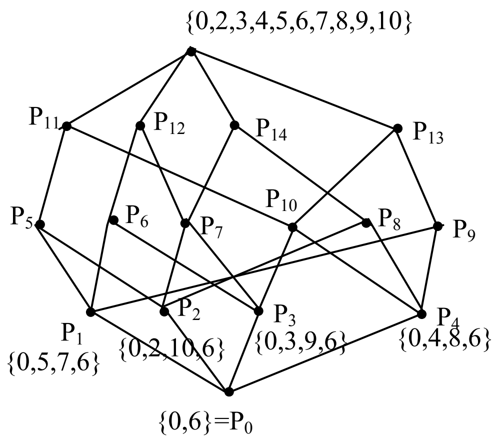
Next we proceed onto define the notion of Smarandache quasi set ideal topological space of a ring (semigroup) over a subring (subsemigroup of the semigroup) of the ring.

DEFINITION 3.10: *Let R be a ring (S a semigroup), R_1 a subring of R (S_1 a subsemigroup of S). T = {collection of all set ideals P of R over the subring R_1 of R such that each P contains a M, M a subring of R (or P is a set ideal of S over the subsemigroup S_1 of S such that each P contains a subsemigroup N_1 of S)}; T is defined as the Smarandache quasi set ideal topological space of the ring (semigroup) over a subring (or over the subsemigroup).*

We will illustrate this situation by some examples.

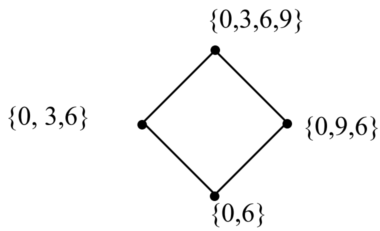
Example 3.79: Let S = Z_{12} be a semigroup under ×.

$S_1 = \{0, 6\}$ be a subsemigroup of S. Let N = {0, 1, 11} be a sub semigroup of S. T = {{0, 6} = P_0, P_1 = {0, 5, 7, 6}, {0, 2, 10, 6} = P_2, P_3 = {0, 3, 9, 6}, P_4 = {0, 4, 8, 6}, {0, 5, 7, 6, 2, 10} = P_5, {0, 5, 7, 6, 3, 9} = P_6, P_8 = {0, 5, 7, 6, 4, 8}, {0, 2, 10, 6, 3, 9} = P_7, P_9 = {0, 2, 10, 6, 4, 8}, P_{10} = {0, 3, 9, 4, 8, 6} P_{11} = {0, 6, 5, 7, 4, 2, 8, 10}, P_{12} = {0, 6, 5, 7, 2, 10, 3, 9}, P_{13} = {0, 6, 2, 10, 3, 9, 4, 8} P_{14} = {0, 6, 5, 7, 4, 8, 3, 9}, P_{15} = {0, 6, 5, 7, 4, 8, 3, 9, 2, 10}} is a Smarandache quasi set ideal topological space of the semigroup S = Z_{12} over the subsemigroup N = {0, 1, 11}. Now T has the following associated lattice which is a Boolean algebra of order 16.



Example 3.80: Let S = Z_{12} be the ring. S_1 = {0, 4, 8} is a subring of S. S_2 = {0, 6} is a subring of S.

Let T = {P_0 = {0, 6}, P_1 = {0, 3, 6}, P_2 = {0, 9, 6}, P_4 = {0, 3, 6, 9}} be the Smarandache quasi set ideal topological space of the ring Z_{12} over the subring {0, 4, 8} = S_1. The lattice associated with T is as follows:



Clearly T is a Boolean algebra of order four.

Now we proceed onto define Smarandache strongly quasi set ideal topological space of a ring (semigroup).

DEFINITION 3.11: *Let S be a ring (semigroup). S_1 and S_2 be two subrings of S; $S_1 \neq S_2$, $S_1 \not\subseteq S_2$, $S_2 \not\subseteq S_1$ (S_1 and S_2 are subsemigroups of S; $S_1 \neq S_2$, $S_1 \not\subseteq S_2$ or $S_2 \not\subseteq S_1$). If T = {collection of all set ideals P of S such that P is a set ideal over both the subrings S_1 and S_2 and $S_1 \subseteq P$ and $S_2 \subseteq P$ (P is a set ideal over the subsemigroups S_1 and S_2 of S and $S_1 \subseteq P$ and $S_2 \subseteq P$)}. T is defined as the Smarandache strongly quasi set ideal topological space of S over subrings S_1 and S_2 (subsemigroups S_1 and S_2) of S.*

We will give examples of this.

Example 3.81: Let S = Z_{12} be the semigroup S_1 = {0, 6} and S_2 = {0, 4} be two subsemigroups of S. Let T = {collection of all Smarandache strongly quasi set ideals P of S over the subsemigroups S_1 and S_2 such that $S_1 \subseteq P$ and $S_2 \subseteq P$} = {P_0 = {0, 4, 6}, P_1 = {0, 4, 6, 3}, P_2 = {0, 4, 6, 8}, P_3 = {0, 4, 6, 9}, {0, 4, 6, 10} = P_4, P_5 = {0, 4, 6, 1}, P_6 = {0, 4, 6, 7}, P_7 = {0, 4, 6, 3, 8}, P_8 = {0, 4, 6, 3, 9}, P_9 = {0, 4, 6, 3, 10}, P_{10} = {0, 4, 6, 3, 1}, P_{11} = {0, 4, 6, 7, 3}, P_{12} = {0, 4, 6, 8, 5}, P_{13} = {0, 4, 6, 8, 11}, P_{14} = {0, 4, 6, 3, 8, 9}, …, {0, 1, 2, 3, …, 11} = Z_{12}}.

Example 3.82: Let R = Z_{15} be the ring of modulo integers. S = {0, 5, 10} and S_1 = {0, 3, 6, 9, 12} be two subrings of R.

Let M = {{0, 5, 10, 3, 6, 9, 12} = P_0, {0, 5, 10, 3, 6, 9, 12, 1} = P_1, P_2 = {0, 5, 10, 3, 6, 9, 12, 2}, P_3 = {0, 5, 10, 3, 6, 9, 12, 4}, P_4 = {0, 5, 10, 3, 6, 9, 12, 7}, P_5 = {0, 5, 10, 3, 6, 9, 12, 8}, P_6 = {0, 5, 10, 3, 6, 9, 12, 11}, P_7 = {0, 5, 10, 3, 6, 9, 12, 13}, P_8 = {0, 5, 10, 6, 3, 9, 12, 14}, …, Z_{15}} be a Smarandache strongly quasi set ideal topological space of the ring R = Z_{15} over the subrings S and S_1.

We also have the associated lattice to be a lattice with P_0 = {0, 5, 10, 3, 9, 6, 12} as the least element and {P_1, P_2, P_3, …, P_8} as atoms.

On similar lines we can define Smarandache perfect set ideal topological space of ring (semigroups) over the subrings (or subsemigroups). This task is left as an exercise to the reader.

We suggest the following problems.

Problems:

1. Find a set ideal of the semigroup

   $$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in Z_2 = \{0, 1\} \right\}.$$

2. Find a Smarandache set ideal of the semigroup

   $$S = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \middle| a, b, d \in Z_3 = \{0, 1, 2\} \right\}.$$

3. Find a S-quasi set ideal of the semigroup

   $$S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a, b, c, d \in Z_3 = \{0, 1, 2\} \right\}.$$

4. Is S given in problem (3) a S-perfect quasi set ideal semigroup?

5. Can $S = Z_2 \times Z_3 \times Z_5$ under component wise multiplication be a S-perfect quasi set ideal semigroup?

6. S(3) be the symmetric semigroup. Is S(3) a S-simple perfect ideal set semigroup?

7. Does the symmetric semigroup S(5) have
   (i) S-set ideal.
   (ii) Set ideal.
   (iii) S-quasi set ideal.

8. S(n) be the symmetric semigroup; n not a prime. Is S(n) a S-perfect quasi set ideal semigroup?

9. Find for the semigroup Z_{124} under multiplication modulo 124.
   (i) S-set ideal.
   (ii) S-quasi set ideal.
   (iii) Set ideal.

10. Can Z_{19} be a S-quasi perfect set ideal semigroup?

11. Obtain some interesting properties about S-perfect quasi set ideal semigroup.

12. Determine the special properties enjoyed by S-simple perfect quasi set ideal semigroups.

13. Let Z_{24} be the semigroup under product.
    (i) Find how many set ideals of Z_{24} exists.
    (ii) Does all subsemigroups of Z_{24} contribute to set ideals of Z_{24}?
    (iii) Does Z_{24} contain a proper subset other than {0} so that subset is a set ideal of Z_{24} for all subsemigroups of Z_{24}?

14. Obtain any other special property associated with set ideals of a semigroup S defined over a subsemigroup S_1 of S.

15. Find all the set ideals of Z_{19} using all subsemigroup of Z_{19} under product?

16. Show Z_p, p a prime, the semigroup under product modulo p has set ideals over subsemigroups of Z_p.

17. Find all the subsemigroups of Z_{29} under $\times$.

18. Let $S = Z_{48}$ be the semigroup, find all set ideals of Z_{48}.
   (i) Does there exist atleast a set ideal of Z_{48} associated with every subsemigroup of Z_{48}?
   (ii) How many set ideals exist for the subsemigroup $S_1 = \{0, 12, 24, 36\}$?
   (iii) Find all the set ideals of Z_{48} associated with the subsemigroup $S_2 = \{0, 3, 6, \ldots, 45\} \subseteq S = Z_{48}$.

19. Let $S = Z_7 \times Z_6$ be a semigroup under product.
   (i) Find all subsemigroups of S.
   (ii) How many set ideals of S exist for the subsemigroup $S_1 = \{0, 1, 6) \times \{0, 1, 5\}$?
   (iii) Does there exist set ideals of S for the subsemigroup $S_2 = \{0, 2, 4, 1\} \times \{0, 1, 3\} \subseteq S$?

20. Find the set ideals associated with S(7), the symmetric semigroup of degree seven.

21. Let $S = \{Z, \times\}$ be the semigroup $S_1 = \{2Z^+ \cup \{0\}\} \subseteq S$ be the subsemigroup of S.
   (i) Find P = {all set ideals of S relative to the subsemigroup S_1}.
   (ii) Is P a set ideal topological space relative to the subsemigroup S_1?
   (iii) If S_1 is replaced by $S_p = \{pZ^+ \cup \{0\} \subseteq S$ $(p > 2)$ find all set ideals of S relative to S_p, the subsemigroup of S.

22. Find some interesting properties associated with S-quasi set ideals of a semigroup S over a subsemigroup S_1 of S.

23. Let S = Z_{28} be the semigroup under product. Can S have S-quasi set ideals over some subsemigroup S_1 of S?

24. Let S = Z_{47} be a semigroup under product.
    (i) Can S have S perfect quasi set ideals?
    (ii) Can S have set ideals?
    (iii) Find the collection of set ideals of Z_{47} over the subsemigroup $S_1 = \{0, 1, 4, 6\} \subseteq S$.

25. Can Z_4 contain S-perfect quasi set ideals?

26. Can Z_9 contain S-perfect quasi set ideals?

27. Find S-perfect quasi set ideals of Z_{40}.

28. Can Z_{625} have S-perfect quasi set ideals?
    (i) Find all set ideals of Z_{625}.

29. Can Z_{186} have set ideals which are not quasi set ideals?

30. Discuss some interesting features enjoyed by set ideals relative to subsemigroups of a semigroup S.

31. Find for the semigroup S = Z_{13} set ideally related subsemigroups.

32. Does there exists a semigroup S which has no subsemigroups which are set ideally related?

33. Does S(5) have subsemigroups which are set ideally related?

34. Discuss the special properties enjoyed by strong set ideals of a semigroup defined over a group.

35. Can S = Z_{28} have strong set ideals of a semigroup defined over a group $G \subseteq S$?

36. Find all the groups of Z_{12}.
Can a subset $P \subseteq Z_{12}$ be both a set ideal over a subsemigroup S_1 as well as a strong set ideal over a group $G \subseteq Z_{12}$, where S_1 is not a S-subsemigroup?

37. Find all the groups of $S = Z_{23}$, the semigroup under $\times$.
(i) How many strong set ideals exists in S?
(ii) How many subsemigroups are in S?
(iii) How many S-subsemigroups are in S?

38. Find all strong set ideals of the semigroup $S = Z_{43}$.

39. Prove every Z_n has a subgroup $n \geq 3$ $\{(Z_n, \times)$ a semigroup$\}$.

40. Obtain some special features enjoyed by special strong set ideals.
Show by an example the difference between these two structures, set ideals and special strong set ideals.

41. Let $S = Z_n$ (n large) be a semigroup under product. Let $G = \{1, n-1\}$ be a group. Does there exist a $P \subseteq S$ so that P is a strong special set ideal of S over G?

42. Obtain some special features enjoyed by two way subsemigroup ideal of a semigroup S.

43. Give an example of a two way subsemigroup ideal of a semigroup.

44. Can the semigroup $S = Z_{47}$ have a two way subsemigroup ideal?

45. Can $S = Z_p$ (p a prime) the semigroup under $\times$ have two way subsemigroup ideal?

46. Find all two way subsemigroup ideals of Z_{36}.

47. Find all two way subsemigroup ideals of Z_{49}.

48. Does there exist a semigroup which has no two way subsemigroup ideals?

49. Can S(6) have two way subsemigroup ideals?

50. Give some special features enjoyed by group-subsemigroup ideal of semigroups.

51. Does S(5) have group-subsemigroup ideals?

52. Does Z_{53} contain group-subsemigroup ideal?

53. Can Z_{128} have group-subsemigroup ideals?

54. Find all group-subsemigroup ideals of Z_{120}.

55. Derive some interesting properties enjoyed by subsemigroup-group ideals of a semigroup S.

56. What is the difference between a subsemigroup-group ideals of a semigroup and group-subsemigroup ideals of the semigroup S?

57. Can a semigroup S have both group-subsemigroup ideal as well as subsemigroup-group ideal?

58. Find group- subsemigroup and subsemigroup-group ideal of a semigroup $S = Z_{420}$.

59. Can S(3) have both group-subsemigroup ideals and subsemigroup-group ideals?

60. Characterize those semigroups Z_n which has both group-subsemigroup ideals and subsemigroup-group ideals?

61. Characterize / does there exists semigroups S which has only group-subsemigroup ideals and does not have subsemigroup-group ideals?

62. Enumerate the special features enjoyed by group-group ideals of a semigroup S.

63. Give examples of group-group ideals of a semigroup S.

64. Find group-group ideals of the semigroup S(9).

65. Can Z_{56} have group-group ideals?

66. Can Z_{72} have group-group ideals?

67. Can Z_{59} have group-group ideals?

68. Does there exists a semigroup which has no group-group ideals?

69. Give an example of a S- subsemigroup-group ideal of a semigroup S.

70. Does Z_{20} have S- subsemigroup-group ideal?

71. Does there exists a semigroup S which has S-subsemigroup-group ideal and does not contain group-S-subsemigroup ideal?

72. Does there exist semigroups which contain both group S-subsemigroups and S- subsemigroup-group ideals?

73. Does a semigroup S which contain both S- subsemigroup-group ideals and group-S- subsemigroup ideals enjoy any other special property?

74. Give an example of a semigroup which has minimal set ideal and maximal set ideal defined over a subsemigroup to be the same.

75. Can $S = Z_{480}$ have minimal set ideals over subsemigroup $S_1 \subseteq S$?

(i) Which of the subsemigroups of S contribute to minimal set ideals?
(ii) Which of the subsemigroups of S contribute to maximal set ideals?
(iii) Does S contain subsemigroup which contribute only to set ideals which are neither maximal nor minimal?
(iv) Study questions (i) to (iii) in case the semigroup $S = Z_n$.

76. Let S = S(10) be the semigroup.
(i) Find all minimal set ideals of S over subsemigroup of S.
(ii) Does S contain maximal set ideals over subsemigroups of S?
(iii) Find group-group set ideals of S(10).
(iv) Can S(10) have group-S- subsemigroup set ideals?
(v) Can S(10) have subsemigroup-group set ideals?
(vi) Can S(10) have two way related set ideals over subsemigroups of S(10)?
(vii) Study questions (1) to (vi) in case of the semigroup S(n) (n an integer).

77. Find the lattice of set ideals of $S = Z_{416}$ over a fixed subsemigroup S_1. Is that lattice a Boolean algebra?

78. Will the lattice of set ideals over a fixed subsemigroup S_1 of a semigroup S be always a Boolean algebra?

79. Let $S = Z_{45}$. Let S_1 be a S- subsemigroup of S.
(i) Find the lattice of set ideals of S relative to the S- subsemigroup over S_1.
(ii) Find the lattice of set ideals of S relative to a subsemigroup S_2 of S where S_2 is not a S- subsemigroup.

80. Will the collection of set ideals of a semigroup S over the same subsemigroup S_1; $S_1 \subseteq S$ will always be a lattice? Justify!

81. Find the lattice L of set ideals of S = Z_{73} over the subsemigroup $S_1 = \{1, 72\}$.
    (i) What is the order of L?
    (ii) Is L a Boolean algebra?
    (iii) If S_1 is replaced by some other subsemigroup S_2 will the order be the same?

82. Obtain some special features enjoyed by the lattice of set ideals of a semigroup S.

83. Find the lattice of set ideals of S(10) where S_1 is the subsemigroup given by

$$S_1 = \left\{\begin{pmatrix} 1 & 2 & 3 & \dots & 10 \\ 1 & 2 & 3 & \dots & 10 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \dots & 10 \\ 2 & 1 & 3 & \dots & 10 \end{pmatrix},\right.$$

$$\left.\begin{pmatrix} 1 & 2 & 3 & \dots & 10 \\ 1 & 1 & 0 & \dots & 10 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \dots & 10 \\ 2 & 2 & 3 & \dots & 10 \end{pmatrix}\right\} \subseteq S(10).$$

84. Is the lattice of set ideals of $(Z, \times)$ for the subsemigroup $S_1 = \{0, -1, 1\}$ a Boolean lattice?

85. Find the structure of the lattice of set ideals of $(Q, \times)$ for the subsemigroup:
    (i) $S_1 = \{0, 1, -1\}$.
    (ii) For the subsemigroup $S_2 = \{3Z\}$.
    (iii) For the subsemigroup $S_3 = \{3Z^+ \cup \{0\}\}$.
    (iv) For the subsemigroup $S_4 = \{10Z^+ \cup \{0\}\}$.

86. Sketch the lattice of set ideals of Z_{243} for the subsemigroup $S_1 = \{0, 1, 242\}$.

87. Describe the lattice of set ideals of $(R, \times)$ for the subsemigroup :
    (i) $S_1 = \{Q, \times\}$.
    (ii) $S_2 = \{Z, \times\}$.
    (iii) $S_3 = \left\{\frac{1}{2^n} \middle| n \in N\right\}$.

88. Let S = (C, ×) be the semigroup. Find the lattice of set ideals of S for the subsemigroup;
    (i) $S_1 = \{1, -1, 0\}$.
    (ii) $S_2 = \{1, -1, 0, i, -i\}$.
    (iii) $S_3 = (Z, \times)$.
    (iv) $S_4 = (Q, \times)$.
    (v) $S_5 = (R, \times)$.
    (vi) $S_6 = \{a + bi \mid a, b \in Z\}$.

89. Let S = $C(Z_{12})$ be the semigroup.
    (i) Find all subsemigroups in S.
    (ii) Find all groups in S.
    (iii) Find set ideals related with subsemigroups.
    (iv) Does S contain group-group set ideals?
    (v) Does S contain S-subsemigroup-group set ideals?
    (vi) Can S contain group-subsemigroup set ideals which are not group-S-subsemigroup set ideals?
    (vii) Study question (i) to (vi) in case $C(Z_n)$; n-a prime, and n a composite number.

90. Let S = $C(Z_{11})$ be the semigroup under product.
    (i) Find groups in S.
    (ii) Find subsemigroup of S.
    (iii) Find S- subsemigroups of S.
    (iv) Find set ideals of S relative to S- subsemigroups.
    (v) Find group-semigroup set ideals of S.

91. What is the special features of topological space of set ideals relative to a subsemigroup?

92. Compare a general topological space with a set ideal topological space relative to a subsemigroup (Z, ×) both defined on Q.

93. How many different topological set ideal spaces can be built using (Z, ×)?

94. Let $S = \langle Z_9 \cup I\rangle$ be a semigroup under $\times$.
    (i) Find set ideals of S related to any subsemigroup S_1 of S.
    (ii) How many subsemigroup does S contain?
    (iii) Does S have group-group set ideals?
    (iv) Can S have S- subsemigroup set ideals?
    (v) Find all groups in S.
    (vi) Study questions (i) to (v) in case of $\langle Z_n \cup I\rangle$, n a composite number.

95. Study questions (i) to (v) in problem 94 in case of $S = \langle Z_{11} \cup I\rangle$.

96. Let $S = \langle C(Z_5) \cup I\rangle$ be a semigroup under product.
    (i) Find all subsemigroups of S.
    (ii) Find set ideals relative to every subsemigroup of S.
    (iii) Find all S- subsemigroups of S.
    (iv) Find all S-set ideals related to every S- subsemigroup of S.
    (v) Find all groups in S.
    (vi) Does S contain group-group set ideals?
    (vii) Does S contain group-S-semigroup set ideals?
    (viii) Does S contain semigroup-group set ideals?

97. Study problem (96) in case of $S = \langle C(Z_p) \cup I\rangle$, p a prime.

98. Let $S = Z_4 \times C(Z_6) \times \langle Z_5 \cup I\rangle = \{(a, b, c) \mid a \in Z_4, b \in C(Z_6), c \in \langle Z_5 \cup I\rangle\}$ be a semigroup under product.
    (i) Find all subsemigroups of S.
    (ii) Find all groups of S.
    (iii) Find all S- subsemigroups of S.
    (iv) Find set ideals of S associated with every subsemigroup.
    (v) Find group-group set ideals of S.

99. Let $S = Z_{12} \times C(Z_8) \times \langle C(Z_{10}) \cup I\rangle$ be the semigroup.
    (i) Find all subsemigroups of S.
    (ii) Find all S- subsemigroups of S.
    (iii) Find all groups in S.

(iv) Relative to every subsemigroup in S find set ideals of S.

100. Let
$S = Z_7 \times Z_{12} \times C(Z_{15}) \times C\ (Z_{16}) \times \langle Z_{11} \cup I\rangle \times \langle C(Z_{20}) \cup I\rangle$
be a semigroup.
(i) Find set ideals of S relative to any subsemigroup.
(ii) If S_1 is any fixed subsemigroup of S find all set ideals of S over S_1.
(a) Is the collection of Boolean lattice?

(b) Find the toplogical space of set ideals associated with it over the subsemigroup S_1 of S.
(c) Find the lattice of set ideals over the subsemigroup S_1 of S and compare the topological space and the lattice of set ideals over S_1 of S.

101. Let $S = \langle C(Z_{49}) \cup I\rangle$ be the semigroup under $\times$.
(i) Find all subsemigroups of S.
(ii) Find all groups in S.
(iii) For the subsemigroup $S_1 = \{0, 1, I\}$ find the set ideals and the lattice of set ideals associated with S_1.
(iv) Let $S_2 = \{0, 1, I, i_F, i_FI, 48, 48I, 48i_F, 48Ii_F\}$ be the subsemigroup of S.

	0	1	I	i_F	i_FI	48	48I	$48i_F$	$48i_FI$
0	0	0	0	0	0	0	0	0	0
1	0	1	I	i_F	i_FI	48	48I	$48i_F$	$48i_FI$
I	0	I	I	Ii_F	Ii_F	48I	48I	$48Ii_F$	$48i_FI$
i_F	0	i_F	Ii_F	48	48I	$48i_F$	$48Ii_F$	1	I
i_FI	0	i_FI	Ii_F	48I	48I	$48i_FI$	$48i_FI$	I	I
48	0	48	48I	$48i_F$	$48i_FI$	1	I	i_F	i_FI
48I	0	48I	48I	$48Ii_F$	$48i_FI$	I	I	Ii_F	i_FI
$48i_F$	0	$48i_F$	$48Ii_F$	1	I	i_F	Ii_F	48	48I
$48i_FI$	0	$48i_FI$	$48i_FI$	I	I	i_FI	i_FI	48I	48I

(v) Find the collection of set ideals of S over S_2.
(vi) Find the lattice associated with the collection of set ideals.

102. Study the properties (i) to (iv) mentioned in problem 101, for $S = \langle C(Z_{p^2}) \cup I \rangle$; p a prime.

103. Derive some interesting properties related with maximal set ideal topological space of a semigroup S (or ring R).

104. Find the maximal set ideal topological space of Z_{120} as a semigroup as well as a ring.

105. Let $S = Z_{42}$ be a semigroup under product.
(i) Find the set ideal topological space of Z_{42} over the subsemigroups
(a) $S_1 = \{0, 1, 41\}$.
(b) $S_2 = \{0, 14, 28\}$.
(c) $S_3 = \{0, 2, 4, \ldots, 40\}$.

106. Find all minimal set ideal topological spaces of the ring $R = Z_{210}$.

107. Let $S = Z_{20}$ be the semigroup.
(i) Can S have a Smarandache quasi set ideal. topological space associated with it?
(ii) Can S be associated with a Smarandache set ideal topological space for a suitable subsemigroup S_1 of S?

108. Study problem (107) when $S = Z_{20}$ is a ring.

Chapter Four

NEW CLASSES OF SET IDEAL TOPOLOGICAL SPACES AND APPLICATIONS

In this chapter we for the first time introduce several new types of topological spaces; using grouprings, semigroup rings, dual number ring, special dual like number ring, special quasi dual number ring of both finite and infinite order. Also algebraic structures like matrices over these rings are used to construct new classes of set ideal topological spaces. We define these new classes of topological spaces of set ideals and illustrate them with examples.

DEFINITION 4.1: *Let Z be the ring of integers. S be a semigroup which is non commutative. ZS be the semigroup ring of the semigroup S over the ring Z. Let I be a subring of ZS. T = {collection of all set right ideals of ZS with respect to the subring I of ZS}.*

We define T as a set right ideal topological space of ZS over the subring I of ZS.

Similarly if V = {collection of all set left ideals of ZS with respect to the subring I of ZS}; we define V to be a set left topological space of ZS over the subring I of ZS.

If ZS is commutative that is a commutative semigroup, the set right ideals coincides with set left ideals that is T = V over the subring I of S.

It is important and interesting to note that we can replace Z in definition 4.1 by R, reals or Q, the rationals or Z_n the modulo integers or C, the complex numbers or $C(Z_n)$ the complex finite modulo integers or $\langle Z \cup I\rangle$ or $\langle Q \cup I\rangle$ or $\langle R \cup I\rangle$ or $\langle C \cup I\rangle$ or $\langle Z_n \cup I\rangle$ or $\langle C(Z_n) \cup I\rangle$ the neutrosophic rings or by ring of dual numbers, or ring of special quasi dual numbers or ring of special dual like numbers and the definition will continue to be true.

We will illustrate this situation by some examples.

Example 4.1: Let S = S(3) be the symmetric semigroup and Z be the ring of integers. ZS the semigroup ring of S over Z.

$$I = \{a + bg \mid g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S(3)\} \subseteq ZS \text{ is a subring}$$

of ZS. Let T = {collection of all right set ideals of ZS over the subring I}. T is a right set ideal topological space of ZS over the subring I of ZS.

Example 4.2: Let S = {all 2 × 2 matrices with entries from Z} be the semigroup and Z_{10} be the ring of integers. $Z_{10}S$ be the semigroup ring. Take any subring I of $Z_{10}S$; using I we can have a left set ideal topological space of $Z_{10}S$ over the subring I.

It is to be noted that the same definition 4.1 holds good for any ring in particular to group rings of a group G over a ring R with unit.

Example 4.3: Let $G = D_{2.7}$ be the dihedral group of order 14. $R = Z_3$ the ring of integers modulo three. RG the group ring. Take $I = \{x + ya \mid x, y \in Z_3, a \in D_{2.7} = \{a, b \mid a^2 = b^7 = 1\ ba = ab^{-1}\}\} \subseteq RG$, the subring of RG.

Let T = {collection of all set right ideals of RG relative to the subring I of RG}; T is a set right ideal topological space of RG over I. Clearly T is of finite order and T has an associated finite lattice.

Example 4.4: Let

$$P = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_6 & a_7 & a_8 \end{pmatrix} \right| a_i \in Z_{40}, 1 \leq i \leq 9 \right\}$$

be the ring of 3 × 3 matrices under usual product of matrices. P is a non commutative ring with unit.

Let

$$M = \left\{ \left. \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_4 & a_5 \\ 0 & 0 & a_6 \end{pmatrix} \right| a_i \in \{0, 2, 4, \ldots, 38\}\ 1 \leq i \leq 6 \right\} \subseteq P$$

be a subring of P. Let T = {all 3 × 3 matrices which are right set ideals of P over the subring M}, T is a right set ideal topological space of P over the subring M of P.

Example 4.5: Let $R = Z_8 A_4$ be the group ring of the group A_4 over the ring Z_8.

Let $S = \{a + bg \mid a, b \in \{0, 2, 4, 6\}$

$$g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \in A_4\}$$

be the subring of R. M = {collection of all set left ideals of R relative to the subring S of R}; M is a left set ideal topological space of R over S.

It is important to mention here that we can construct infinitely many left and right set ideal topological spaces over subrings using grouprings or semigroup rings or square matrix rings finite or infinite order.

Next we can using dual number rings, special dual like number rings, special quasi dual number rings and all types of mixed dual numbers get the set ideal topological spaces.

Infact these rings can be replaced by semigroups and the corresponding set ideal topological spaces can be obtained.

Infact these new topological space of set ideals may have several interesting properties.

We will illustrate these concepts by some examples.

Example 4.6: Let

$$R = \{a + bg \mid a, b \in Z_{45}; g^2 = 0, g \text{ a new element}\}$$

be the dual number ring. Let

$S = \{a + bg \mid a, b \in \{0, 15, 30\}$; g the new element such that $g^2 = g\}\} \subseteq R$ be a subring of R.

T = {collection of all set ideals of R over the subring S of R}; T is set ideal topological space of dual numbers of R over the subring S.

Example 4.7: Let

$R = Z (g_1, g_2) = \{a + bg_1 + cg_2 \mid a, b, c \in Z, g_1 = 4, g_2 = 8 \in Z_{16}\}$ be the dual number ring of dimension three over Z.

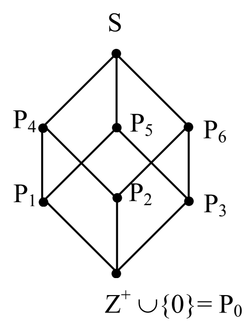
Let $S = Z \subseteq R$ be the subring of R. T = {collection of all Smarandache set ideals of R over the subring Z or R}. T is a Smarandache set ideal topological space of R over the subring S of R.

Example 4.8: Let $S = (Z^+ \cup \{0\}) (g_1, g_2, g_3) = \{a + bg_1 + cg_2 + dg_3 \mid a, b, c, d \in Z^+ \cup \{0\}; g_1 = 8, g_2 = 16 \text{ and } g_3 = 24; g_i \in Z_{32}, 1 \leq i \leq 3\}$ be a four dimensional dual number semigroup.

Let $S_1 = Z^+ \cup \{0\} \subseteq S$ be a subsemigroup of S.

T = {collection of all Smarandache set ideals of S relative to the subsemigroup S_1 of S}; T is a Smarandache set ideal dual number topological space of S over the subsemigroup S_1.

$T = \{P_0 = Z^+ \cup \{0\}, P_1 = \{a + bg_1 \mid a, b \in Z^+ \cup \{0\}, g_1 = 8\} \subseteq S, P_2 = \{a + bg_2 \mid a, b \in Z^+ \cup \{0\}, g_2 = 16\} \subseteq S, P_3 = \{a + bg_3 \mid a, b \in Z^+ \cup \{0\}; g_3 = 24\} \subseteq S, P_4 = \{a + bg_1 + cg_2 \mid a, b, c \in Z^+ \cup \{0\}, g_1 = 8 \text{ and } g_2 = 16\} \subseteq S, P_5 = \{a + bg_1 + cg_3 \mid a, b, c \in Z^+ \cup \{0\}, g_1 = 8 \text{ and } g_3 = 24\} \subseteq S, P_6 = \{a + bg_2 + cg_3 \mid a, b, c \in Z^+ \cup \{0\}, g_2 = 16, g_3 = 24\} \subseteq S \text{ and } P_7 = (Z^+ \cup \{0\}) (g_1, g_2, g_3) = S\}$. The lattice of Smarandache dual number set ideals of S over the subsemigroup S_1 is as follows:



Clearly the lattice is a Boolean algebra of S-dual number set ideals of the semigroup over the subsemigroup S_1 of S.

Example 4.9: Let $S = C(Z_2) (g_1, g_2) = \{(a + bi_F) + (c+di_F)g_1 + (e+fi_F) g_2 \mid a, b, c, d, e, f \in Z_2, g_1 = 6 \text{ and } g_2 = 12 \in Z_{36}\}$ be the semigroup of dual number of dimension three under product.

$S = \{0, 1, g_1, g_2, g_1 + g_2, i_F, i_Fg_1, i_Fg_2, g_1+i_Fg_1, g_2 + i_Fg_2, g_1+i_Fg_2, 1+ g_1+ g_2, \ldots, 1+ i_F + g_1 + g_2 + i_Fg_1 + i_Fg_2\}$.

Let $S_1 = \{C(Z_2)g_1\} \subseteq S$ be the subsemigroup of S. T = {collection of all Smarandache set ideals of S over the subsemigroup S_1 of S}; T is a Smarandache set ideal three dimensional dual number topological space of S over the subsemigroup S_1 of S.

If S is considered as a ring then S_1 is the subring of S and T is a Smarandache set ideal three dimensional dual number topological space of S over the subring S_1 of S.

Example 4.10: Let $S = \{(\langle C(Z_7) \cup I\rangle) (g_1, g_2, g_3, g_4) = a + bg_1 + cg_2 + dg_3 + eg_4 \mid a, b, c, d, e \in \langle C(Z_7) \cup I\rangle; g_1 = 9, g_2 = 18, g_3 = 27 \text{ and } g_4 = 36 \in Z_{81}\}$ be the semigroup of five dimensional dual numbers.

Let $S_1 = \{a + bg_1 + cg_2 \mid a, b, c \in Z_7, g_1 = 9 \text{ and } g_2 = 18\} \subseteq S$ be the subsemigroup of semigroup S. T = {collection of all Smarandache set ideals of S over the subsemigroup S_1 of S}. T is a Smarandache set ideal five dimensional dual number topological space of S over the subsemigroup S_1 of S.

S is also a ring of dual numbers of dimension five with complex neutrosophic coefficients. S_1 is also a subring of dimension three with integer coefficients.

Thus T is also a Smarandache set ideal five dimensional dual number topological space of the ring over the subring S_1 of S.

We see T is both a topological space over rings as well as topological space over the semigroup.

Example 4.11: Let $S = \{\langle Z_{12} \cup I\rangle (g_1, g_2, g_3) \mid a + bg_1 + cg_2 + dg_3$ where $a, b, c, d \in \langle Z_{12} \cup I\rangle, g_1 = 8, g_2 = 16 \text{ and } g_3 = 24 \in Z_{64}\}$ be a four dimensional neutrosophic ring (semigroup) of dual numbers.

Take $S_1 = \{a + bg_1 + cg_2 + dg_3 \mid a, b, c, d \in \langle\{0, 2, 4, 6, 8, 10\} \cup I\rangle; g_1 = 8, g_2 = 16$ and $g_3 = 24 \in Z_{64}\} \subseteq S$ to be the neutrosophic subring (neutrosophic subsemigroup) of S.

Let T = {collection of Smarandache set ideals of S over the subring S_1 (or subsemigroup S_1) of S}; T is a four dimensional set ideal neutrosophic topological space of dual numbers of S over the subring S_1 (or subsemigroup S_1) of S.

Example 4.12: Let $S = \{\langle Z \cup I\rangle (g_1, g_2, g_3, g_4, g_5) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5$ where $a_j \in \langle Z \cup I\rangle$; $1 \leq j \leq 6$, $g_1 = 8$, $g_2 = 16$, $g_3 = 24$, $g_4 = 32$, $g_5 = 40 \in Z_{64}\}$ be the semigroup of neutrosophic dual numbers of dimension six.

$S_1 = \{\langle Z \cup I\rangle (g_1, g_2) = a_1 + a_2g_1 + a_3g_3 \mid a_j \in \langle Z \cup I\rangle; 1 \leq j \leq 3$, $g_1 = 8$ and $g_2 = 16 \in Z_{64}\} \subseteq S$ be a subsemigroup of S.

T = {collection of all Smarandache set ideals of S over the subemigroup S_1 of S}; T is a neutrosophic six dimensional dual number topological space of set ideals of S over the subsemigroup S_1 of S.

Now having seen topological higher dimensional set ideal space of dual numbers we proceed onto give examples of special dual like number of set ideal topological spaces.

Example 4.13: Let $S = \{a + bg \mid a, b \in Z_{42}, g = 4 \in Z_{12}\}$ be ring of special dual like numbers.

$S_1 = \{a + bg \mid a, b \in \{0, 7, 14, 21, 28, 35\}, g = 4 \in Z_{12}\} \subseteq S$ be a subring of special dual like numbers of the ring S. T = {collection of all Smarandache set ideals of S relative to the subring S_1 of S}; T is defined as the Smarandache set ideal topological space of special dual like numbers of S over the subring S_1 of S.

Example 4.14: Let $S = \{C (Z_7) (g_1, g_2) = a_1 + a_2g_1 + a_3g_2$ where $a_j \in C(Z_7)$; $1 \leq j \leq 3$; $g_1 = 4$ and $g_2 = 9 \in Z_{12}\}$ be the special dual like number of dimension three semigroup.

$S_1 = \{a_1 + a_2g_1 \mid a_1, a_2 \in Z_7, g_1 = 4 \in Z_{12}\} \subseteq S$ be a subsemigroup of S. T = {collection of Smarandache set ideals of S over the subsemigroup S_1 of S}.

T is the Smarandache set ideal topological space of special dual like numbers of S over the subsemigroup S_1.

We can also have the lattice of S-set ideals of S associated with T.

Example 4.15: Let $S = \{(Q^+ \cup \{0\})\ (g_1, g_2, g_3, g_4) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4$ with $a_j \in Q^+ \cup \{0\};\ g_i^2 = 0, g_ig_j = 0, i \neq j, 1 \leq i, j \leq 4\}$ be a five dimensional semigroup of dual numbers.

Let $I = \{(Q^+ \cup \{0\})\ (g_1) = a + bg_1$ where $a, b \in Q^+ \cup \{0\}, g_1^2 = 0\} \subseteq S$ be a subsemigroup. T = {Collection of all Smarandache set ideals of S over the subsemigroup I}; T is a Smarandache five dimensional dual number set ideal topological space of S over the subsemigroup I of S.

Example 4.16: Let $\{(Z^+ \cup \{0\})\ (g_1, g_2, g_3, g_4, g_5, g_6) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4 + a_6g_5 + a_7g_6$ with $a_j \in Z^+ \cup \{0\}, 1 \leq j \leq 7,$

$$g_1 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 \\ 0 \\ I \\ 0 \\ 0 \\ 0 \end{bmatrix}, g_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ I \\ 0 \\ 0 \end{bmatrix}, g_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I \\ 0 \end{bmatrix} \text{ and } g_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ I \end{bmatrix}$$

we see $g_i \times_n g_i = g_i^2;\ 1 \leq i \leq 6\} = S$ be the semigroup under $\times$.

Take

$$S_1 = \{(Z^+ \cup \{0\})\ (g_1) = a + bg_1 \mid a, b \in Z^+ \cup \{0\},$$

$$g_1 = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} \subseteq S \text{ be a subsemigroup of S.}$$

T = {collection of all Smarandache set ideals of S over S_1}; T is a Smarandache set ideal topological space of special quasi dual numbers of S over S_1 basic set ideal set as {$P_1 = Z^+ \cup \{0\}$ (g_1, g_2), $P_2 = (Z^+ \cup \{0\})$ (g_1, g_3), $P_3 = Z^+ \cup \{0\}$ (g_1, g_4), $P_4 = Z^+ \cup \{0\}$ (g_1, g_5) and $P_5 = Z^+ \cup \{0\}$ (g_1, g_6)}.

We see this topological space has a Boolean algebra representation with least element $(Z^+ \cup \{0\} (g_1)) = S_1$ and atoms P_1, P_2, P_3, P_4 and P_5 with 2^5 elements in it.

Example 4.17: Let R = $\{\langle Z_{25} \cup I\rangle (g_1, g_2, g_3, g_4) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4$ where $a_j \in \langle Z_{25} \cup I\rangle$, $1 \leq j \leq 5$;

$$g_1 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, g_2 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, g_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, g_4 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix};$$

$$g_i \times_n g_i = g_i, 1 \leq i \leq 4, g_i \times_n g_j = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; 1 \leq i, j \leq 4\}$$

be the ring of special dual like numbers.

$R_1 = \{a_1 + a_2g_3 + a_3g_4 \mid a_i \in \{0, 5, 10, 15\}; 1 \leq i \leq 3,$

$$\left. g_3 = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix} \text{ and } g_4 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \right\} \subseteq R$$

be a subring of R.

T = {all Smarandache set ideals of R over the subring R_1 of R}; T is a Smarandache set ideal topological space of the ring R of special dual like numbers of dimension five over the subring R_1 of R.

Example 4.18: Let S = $\{Z_6 (g_1, g_2, g_3, g_4, g_5) = a_1 + a_2g_1 + \ldots + a_6g_5$ with $a_i \in Z_6$; $1 \leq i \leq 6$, $g_1 = 4$, $g_2 = 6$, $g_3 = 8$, $g_4 = 9$, $g_5 = 3 \in Z_{12}\}$ be the semigroup of special mixed dual number of dimension six.

Let $S_1 = Z_6 (g_2)$ be a subsemigroup of S.

T = {collection of Smarandache set ideals of S over the subring S_1 of S}, T is a S-set ideal topological space of special mixed dual numbers of S over S_1.

Now having seen set ideal topological spaces of special dual like numbers, special quasi dual numbers, dual numbers and then mixed structures we now proceed onto give examples of set ideal topological spaces of semigroups (or rings) build using matrices under natural product.

Example 4.19: Let

$$S = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right| a_i \in \langle Z \cup I\rangle; 1 \leq i \leq 4\right\}$$

be a ring under natural product $\times_n$.

$$S_1 = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}\right| a_i \in Z\right\} \subseteq S$$

be a subring P = {all S-set ideals of S over the subring S_1}. P is a S-set ideal topological space of column matrices under natural product of S over S_1.

Example 4.20: Let

S = {all 3×3 matrices with entries from Z_{48}} be a non commutative ring under usual product.

S_1 = {all 3×3 matrices with entries from $\{0, 12, 24, 36\} \subseteq Z_{48}$} $\subseteq$ S be a subring of S.

Let T = {collection of all set ideals of S over the subring S_1 of S}, T is a set ideal 3×3 square matrices, topological space of S over S_1.

Example 4.21: Let
R = {collection of all 3×5 matrices with entries from $\langle Z_{12} \cup I\rangle$} be the ring under natural product $\times_n$.

R_1 = {collection of all 3×5 matrices with entries from $\langle\{0, 6\} \cup I\rangle$} $\subseteq$ R be a subring of R.

T = {collection of all S-set ideals of R over the subring R_1}; T is a S-set ideal topological space of the ring R of 3×5 matrices under natural product over the subring R_1 of R.

Example 4.22: Let P = {all 4×2 matrices with entries from $C(Z_{18})$ (g_1, g_2, g_3, g_4) = $\{a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4$ where $a_j \in C(Z_{18})$; $g_1 = 6$, $g_2 = 4$, $g_3 = 9$ and $g_4 = 8 \in Z_{12}\}$, $1 \leq j \leq 4\}\}$ be the ring of 4×2 matrices with special mixed dual number entries under the natural product $\times_n$.

Let S_1 = {all 4×2 matrices with entries from M = $\{a_1 + a_2g_1 \mid a_1, a_2 \in \{0, 6, 12\}, g_1 = 4\}$ $\subseteq$ P be the subring of P. T = {collection of all set ideals of P over the subring S_1 of P}; T is a set ideal topological space of set ideals of matrix special mixed dual numbers of P over S_1.

Now we briefly mention a few of the applications of these new set ideal topological space of semigroups and rings over subsemigroups and subrings respectively.

They can be applied in places where topological spaces find applications. Infact these new structures can also find applications in places were constraints are used. So that they have the liberty to use the ring / semigroup structure and these set ideals over a suitable subring / subsemigroup and use them in the place of usual topological spaces.

Also one important and relevant problem is that does these new finite topological spaces contribute to more number of finite topological spaces which already exist?

Problem:

1. Describe some special features enjoyed by the topological spaces of special dual like numbers build using the ring $\langle C(Z_n) \cup I\rangle$ (g_1, g_2).

2. Describe some special features enjoyed by the topological space of special quasi dual number of dimension five semigroup $(Z^+ \cup \{0\})$ (g_1, g_2, g_3, g_4).

3. Let $S = \{Z_9 (g_1, g_2, g_3) = a_1 + a_2g_1 + a_3g_2 + a_4g_3$ where $a_j \in Z_9$, $1 \leq i \leq 4$; $g_1 = (1, 1, 0, 0, 0)$, $g_2 = (0, 0, 1, 0, 0)$ and $g_3 = (0, 0, 0, 1, 1)\}$ be the semigroup of special dual like numbers over the semigroup S.
   Let $S_1 = \{a_1 + a_2g_1 \mid a_1, a_2 \in \{0, 3, 6\}; g_1 = (1, 1, 0, 0, 0)\} \subseteq S$, be the subsemigroup of S. Let T be the set ideal topological space of special dual like numbers over the subsemigroup S_1.

   (i) Find a basic set of T.
   (ii) Give the lattice representation L of T.
   (iii) Is L a Boolean algebra?

(iv) If in T we make the collection of T_1 set ideals of topological space as the Smarandache set ideals a topological space of special dual like numbers, what is the difference between T and T_1?
(v) Find the lattice associated with T_1.
(vi) Compare the lattices T and T_1.

4. Study problem (3) if S is realized as a ring.

5. Let $S = \langle Z_8 \cup I\rangle (g_1, g_2, g_3, g_4) = a_1 + a_2g_1 + a_3g_2 + a_4g_3 + a_5g_4$ where $a_i \in \langle Z_8 \cup I\rangle$; $1 \leq i \leq 5$, $g_1 = (–I, 0, 0, 0)$, $(0, –1, 0, 0) = g_2$, $g_3 = (0, 0, –I, 0)$ and $g_4 = (0, 0, 0, –1)$ where $g_j^2 = –g_j$, $1 \leq j \leq 4$; $g_ig_j = (0)$ if $i \neq j$, $1 \leq i, j \leq 4\}$ be the ring of special quasi dual numbers.
Let $S_1 = \{Z_8 (g_1, g_2) = a_1 + a_2g_1 + a_3g_2 \mid a_i \in Z_8; 1 \leq i \leq 8; g_1 = (–I, 0, 0, 0), g_2 = (0, –1, 0, 0)\} \subseteq S$ be a subring of S. T = {collection of all S-set ideals of S over the subring S}; T is a S-set ideal topological space of special quasi dual numbers of S over the subring S_1 of S.

(i) Find the number of elements in T.
(ii) Find a basic set of T.
(iii) Find the lattice L associated with T.
(iv) Is L a Boolean algebra?
(v) Can T have non trivial topological subspaces?

6. Let $S = \left\{\left.\begin{bmatrix} a_1 & a_2 & ... & a_6 \\ a_7 & a_8 & ... & a_{12} \\ a_{13} & a_{14} & ... & a_{18} \end{bmatrix}\right| a_i \in Z_{20}; 1 \leq i \leq 18\right\}$

be a ring under natural product.

Let

$S_1 = \left\{\left.\begin{bmatrix} a_1 & 0 & 0 & 0 & 0 & a_2 \\ 0 & a_3 & 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 & 0 & 0 \end{bmatrix}\right| a_i \in Z_{20}; 1 \leq i \leq 6\right\} \subseteq S$

be a subring of S. Let T = {collection of all set ideals of S over the subring S_1 of S}. T is a set ideal topological space of the ring S over the subring S_1.

(i) Find a basic set of T.
(ii) Obtain the lattice L associated with T.
(iii) Can T have subspaces?
(iv) If T_1 is the collection of S-set ideals of S over S_1; that is T is S-set ideal topological space of S over S_1. Compare T and T_1 as topological spaces.

7. Let S = $\{(Z_8 \times Z_7 \times Z_5 \times Z_{12})\}$ be the semigroup. P = $\{(a, 0, 0, b) \mid a \in 2Z_4$, and $b \in 3Z_{12}\} \subseteq$ S be a subsemigroup of S.

(i) Find set ideal topological space T of S over P.
(ii) Find a basic set of T.
(iii) Can T have subspaces?
(iv) Find the lattice L associated with T.

8. Find some nice application of set ideal topological spaces of a ring R over a subring.

9. Suppose T is a S-set ideal topological space of a semigroup S over a subsemigroup S_1 of S. L the lattice associated with T.

(i) When will L be a Boolean algebra?
(ii) Can L be a modular lattice which is not a distributive lattice?
(iii) Is it possible to associate with every sublattice of L a set ideal topological subspace of T and vice versa?

10. Obtain some special applications of set ideal topological spaces of a semigroup S of mixed special dual numbers over a subsemigroup S_1 of S.

11. Bring out the difference between a topological space with 2^N elements and set ideal topological space of order 2^N.

12. Distinguish between the set ideal topological space over a ring and over a semigroup.

13. If Z_n is taken, n any composite number, can Z_n have the same set ideal topological space as a ring and as well as a semigroup?

14. Let P = $\{C(Z_{40}), \times\}$ be the semigroup.

    Let S = $\{0, 10, 20, 30\} \subseteq$ P be a subsemigroup of P.

    (i) Find the set ideal topological space T of P over the subsemigroup S of P.
    (ii) Find the number of elements T.
    (iii) Find the lattice associated with T.
    (iv) Find the Smarandache set ideal topological space T_1 of P over the subsemigroup S of P.
    (v) Find the basic set of T_1.
    (vi) Find the number of elements in T_1.
    (vii) Find the lattice L_1 associated with T_1.
    (viii) Can the lattice L_1 be a Boolean algebra?

15. Let S = Z_{424} be the ring. Let $S_1 = \{0, 4, \ldots, 420\} \subseteq$ S be a subring of S. Study problems (i) to (viii) mentioned in problem 14 in case of this S.

16. Let S = Q $(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \sqrt{13})$ be a field. S_1 = Q($\sqrt{2}$) be a subring.
    T = {collection of all S-set ideals of S over the subring S_1}.

    (i) Is T a topological space of S-set ideals?
    (ii) Is it finite or infinite?
    (iii) Can T have a lattice representation? Justify!

17. Let Z_{30} = S be the ring. $S_1 = \{0, 2, \ldots, 28\}$ a subring of S.

    (i) Can S have a set ideal topological space of S with respect S_1?
    (ii) Take $S_2 = \{0, 15\}$. Find the set ideal topological space of S over S_2.
    (iii) Can these set ideal topological spaces have associated lattices?
    (iv) Are these lattices Boolean algebras?

18. Give examples of special set ring. (Recall; we define a ring R to be a special set ring if R has no S-set ideals only set ideals)

19. Can Z_{428} be a special set ring?

20. Can Z_{13} be a special set ring?

21. Let S = $Z_2[x]$ be a ring I = $\langle x^4 + 1\rangle$ be an ideal of S. T = $x^2[x] \mid I$ be a ring.
    (i) Can P = $\{I, 1 + x + x^2 + x^3 + I\} \subseteq T$ be a S-set ideal of T?
    (ii) Can we built S-set ideal topological spaces on T?

22. Define maximal S-set ideal of a ring and give some examples.

23. Prove a maximal set ideal of a ring R in general need not be a S-set maximal ideal of R.

24. Can Z_{18} have S-maximal set ideal?

25. Define S-set minimal ideal and provide some examples.

26. Can ring R have I to be both a S-set maximal ideal as well as a S-set minimal ideal?

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book the authors use set ideals of rings (or semigroups) to build topological spaces. These spaces are dependent on the set over which the set ideals are defined. It is left as an open problem whether this newly constructed topological space of finite order increases the existing number of finite topological spaces.

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